Analysis of Diffraction by three semi-infinite planes using Wiener-Hopf method

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Abstract

The diffraction of an incident plane wave on three equally spaced half-planes is formulated in terms of Wiener-Hopf (W-H) equations using an equivalent circuit model. The order of matrix factorization is reduced due to the symmetry of the problem. A new solution of this classic Wiener-Hopf problem is proposed and the formulation of the problem is presented for the general case of skew incident plane wave.

1. Formulation

The well-known in literature diffraction problem of three equally spaced semi-infinite PEC planes [1] is formulated in terms of Wiener-Hopf (W-H) equations. We consider the general case of an arbitrary plane wave incident (1) on the structure:

$$\begin{cases} \boldsymbol{E}_z^i = \boldsymbol{E}_o \boldsymbol{e}^{j\tau_o \rho \cos(\varphi-\varphi_o\}} \boldsymbol{e}^{-j\alpha_o z} \\ \boldsymbol{H}_z^i = \boldsymbol{H}_o \boldsymbol{e}^{j\tau_o \rho \cos(\varphi-\varphi_o)} \boldsymbol{e}^{-j\alpha_o z} \end{cases} \tag{1}$$

where k is the free-space wave number, $\tau_o = k \sin\beta$, $\alpha_o = k\cos\beta$, and β and φ_o are respectively the zenithal and azimuthal angles of the incident wave. Figure 1 illustrates the geometry of the problem for an E-polarized normal incident plane wave ($\beta = \pi/2$).

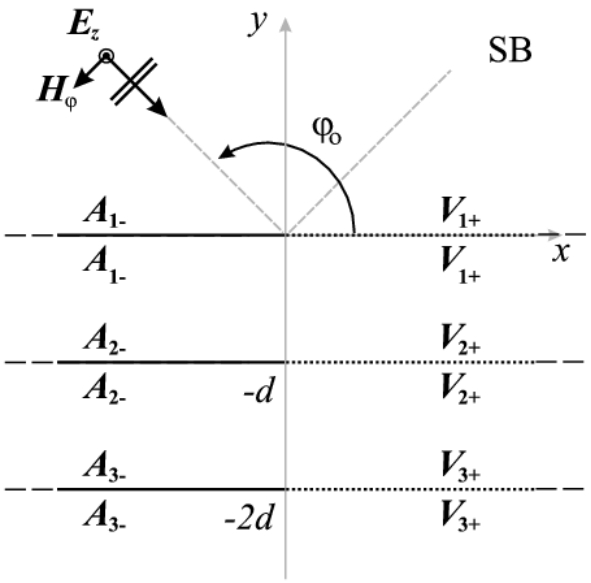

Figure 1: Diffraction on three equally spaced semi-infinite PEC planes: E-polarization.
The figure reports the Wiener-Hopf unknowns.

In order to formulate the problem in the spectral domain we apply the Fourier transform to the transverse components of the electromagnetic fields (2):

$$\begin{aligned}\mathbf{V}(\eta, y) &= e^{j\alpha_o z}\int_{-\infty}^{\infty} \hat{y}\times\mathbf{E}_t(x,y,z)e^{j\eta x}dx \\ \mathbf{I}(\eta, y) &= e^{j\alpha_o z}\int_{-\infty}^{\infty} \mathbf{H}_t(x,y,z)e^{j\eta x}dx\end{aligned} \tag{2}$$

We define an equivalent circuit problem, see Figure 2, where the characteristic admittance of the free space is a matrix [2]:

$$\mathbf{Y_c} = \frac{Y_o}{k\sqrt{\tau_O^2-\eta^2}}\begin{vmatrix}\tau_O^2 & -\eta\,\alpha_O \\ -\eta\,\alpha_O & k^2-\eta^2\end{vmatrix} \tag{3}$$

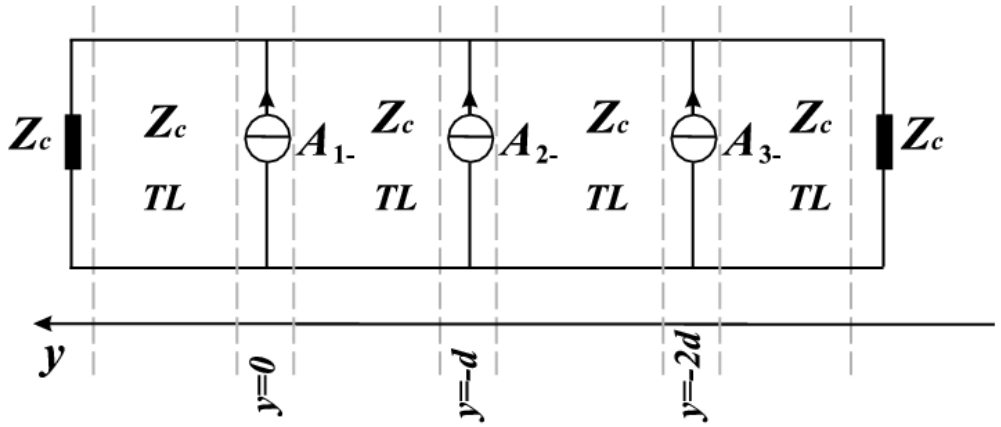

Figure 2: Equivalent Circuit model of the problem under study.

The equivalent circuit models the propagation inside the region defined by the half-planes (region 1, 2 and 3) using a Π model of matrix admittance:

$$\mathbf{Y}_1 = \mathbf{Y}_2 = j\mathbf{Y}_c\tan\frac{\tau d}{2},\ \mathbf{Y}_3 = -j\mathbf{Y}_c\frac{1}{\sin\tau d} \tag{4}$$

where $\tau = \sqrt{\tau_o^2-\eta^2}$.

The circuit nodal analysis provide the three following equations

$$\begin{aligned}&\text{node } 1 \Rightarrow \quad (\boldsymbol{Y}_c+\boldsymbol{Y}_1+\boldsymbol{Y}_3)\boldsymbol{V}_{1+}-\boldsymbol{Y}_3\boldsymbol{V}_{2+} = \boldsymbol{A}_{1-} \\ &\text{node } 3 \Rightarrow \quad (\boldsymbol{Y}_c+\boldsymbol{Y}_1+\boldsymbol{Y}_3)\boldsymbol{V}_{3+}-\boldsymbol{Y}_3\boldsymbol{V}_{2+} = \boldsymbol{A}_{3-} \\ &\text{node } 2 \Rightarrow \quad 2(\boldsymbol{Y}_1+\boldsymbol{Y}_3)\boldsymbol{V}_{2+}-\boldsymbol{Y}_3\boldsymbol{V}_{1+}-\boldsymbol{Y}_3\boldsymbol{V}_{3+} = \boldsymbol{A}_{2-}\end{aligned} \tag{5}$$

where $\boldsymbol{V}_{1+}, \boldsymbol{V}_{2+}$ and $\boldsymbol{V}_{3+}$ and $\hat{\boldsymbol{y}}\times\boldsymbol{A}_{1-}, \hat{\boldsymbol{y}}\times\boldsymbol{A}_{2-}$ and $\hat{\boldsymbol{y}}\times\boldsymbol{A}_{3-}$ are respectively the Fourier transforms of the transverse electric field on the three apertures $y = 0, -d, -2d$ and the total currents induced on the three half- planes. The previous equations constitute a W-H system of third order where the unknowns and the admittance are respectively vector and matrix of size 2, therefore the W-H problem is a scalar matrix problem of size 6.

The symmetry of the geometry allows to reduce the order of the vector W-H problem. In fact by summing the first two equations we get the following system of equations:

$$\begin{cases} \left(\boldsymbol{Y_c + Y_1 + Y_3}\right)\boldsymbol{(V_{1+} + V_{3+})} - \boldsymbol{2Y_3 V_{2+}} = \boldsymbol{A_{1-} + A_{3-}} \\ \boldsymbol{-Y_3 (V_{1+} + V_{3+})} + \boldsymbol{2}\left(\boldsymbol{Y_1 + Y_3}\right)\boldsymbol{V_{2+}} = \boldsymbol{A_{2-}} \end{cases} \tag{6}$$

Equation (6) is a homogeneous vector W-H problem of order 2 ($\boldsymbol{G \cdot V_+ = A_-}$), where the plus unknowns are $\boldsymbol{(V_{1+} + V_{3+})}$ and $\boldsymbol{2V_{2+}}$, the minus unknowns are $\boldsymbol{(A_{1-} + A_{3-})}$ and $\boldsymbol{A_{2-}}$ and the kernel takes the following form:

$$\boldsymbol{G} = \begin{vmatrix} \left(\boldsymbol{Y_c + Y_1 + Y_3}\right) & \boldsymbol{-Y_3} \\ \boldsymbol{-Y_3} & \left(\boldsymbol{Y_1 + Y_3}\right) \end{vmatrix} = \boldsymbol{Y_c \otimes H} \tag{7}$$

where the operator $\otimes$ is the Kronecker matrix product and where

$$\boldsymbol{H} = \begin{vmatrix} 1 - j\,Cot(\tau d) & j\,Csc(\tau d) \\ j\,Csc(\tau d) & -j\,Cot(\tau d) \end{vmatrix} = \frac{2}{1 - e^{-2j\tau d}} \begin{vmatrix} 1 & -e^{-j\tau d} \\ -e^{-j\tau d} & \left(\dfrac{1 + e^{-j\tau d}}{2}\right) \end{vmatrix} \tag{8}$$

2. Solution

In order to obtain the solution of the Wiener-Hopf problem we need to factorize (7).
By applying the mixed-product property of the Kronecker product (9) to the plus and minus factorized matrices of $\boldsymbol{Y_c}$ and $\boldsymbol{H}$ we obtain the factorization of $\boldsymbol{G}$, see (10).

$$A \cdot C \otimes B \cdot D = A \otimes B \cdot C \otimes D \tag{9}$$

$$\boldsymbol{G = G_- \cdot G_+ = Y_{c-} \cdot Y_{c+} \otimes H_- \cdot H_+ = Y_{c-} \otimes H_- \cdot Y_{c+} \otimes H_+} \tag{10}$$

From the homogenous Wiener-Hopf equation (6) we obtain the normal form ($G(\alpha)F_+(\alpha) = F_-(\alpha) + F_{o+}(\alpha)$) by extracting the source term related to the incident plane wave:

$$\boldsymbol{G \cdot V_+ = A_-^d + \frac{R_o}{\eta - \eta_o}} \tag{11}$$

$\boldsymbol{R_o}$ is the source obtained as GO component from the incident field.
We remind the spectral quantities are functions of the spectral variable η. The solution takes the following form:

$$\begin{cases} \boldsymbol{V_+(\eta) = G_+^{-1}(\eta) \cdot G_+^{-1}(\eta_o) \cdot \frac{R_o}{\eta - \eta_o}} \\ \boldsymbol{A_-(\eta) = G\ (\eta) \cdot V_+(\eta) = G_-(\eta) \cdot G_+^{-1}(\eta_o) \cdot \frac{R_o}{\eta - \eta_o}} \end{cases} \tag{11}$$

We note that the matrix $\boldsymbol{Y}_c$ can be factorized in closed form. However no exact factorization has been obtained for matrices $\boldsymbol{G}$ and $\boldsymbol{H}$ up to now, therefore we can apply the approximate procedure presented in [3, 4] based on integral equations.
Numerical results will be presented at the Conference.

3. Acknowledgments

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4. References

1. D.S. Jones, "Diffraction by three semi-infinite planes," *Proc. Royal Soc. London,* **A404**, 1986, pp. 299–321.

2. V. Daniele, "Electromagnetic propagation in plane stratified regions," Internal Report ELT-2006, Politecnico di Torino, 2006, available at http://www.eln.polito.it/staff/daniele.

3. V. Daniele and G. Lombardi, "Wiener-Hopf solution for impenetrable wedges at skew incidence," *IEEE Trans. Antennas Propag.*, **AP-54**, 2006, pp. 2472-2485, doi:10.1109/TAP.2006.880723.

4. V. Daniele, and G. Lombardi, "Fredholm factorization of Wiener-Hopf scalar and matrix kernels," *Radio Science*, **42**, 2007, RS6S01, doi:10.1029/2007RS003673.