On the Scattering of a Plane Wave by a Transparent Wedge

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2-D scalar problem of the scattering of plane wave by a transparent wedge is considered. Dependence on time is harmonic. The velocities of the wave propagations inside of the wedge and outside of it are not equal. Boundary conditions on the interface are classical.

If some condition of nondegeneracy is valid, existence and uniqueness theorems are proved. The statement of the problem is traditional: as inside, as well outside of the wedge the corresponding wave equation are satisfied, the Meixner condition takes place in a neighborhood of the vertex of the wedge, if to remove from the solution the incident wave, all reflected and refracted waves, the remaining part of the solution satisfies the radiation condition in integral form.

1. Formulation of the problem

Let Ω_i be domain $0 < \varphi < \Phi,\ r > 0,\ \Phi = \text{const},\ 0 < \Phi < 2\pi$, (here r, φ – is the classical polar coordinate system) and $\Omega_o = \mathbb{R}^2 \backslash \bar{\Omega}_i$ – is external domain with respect to Ω_i: $\Omega_o = \{r, \varphi :\ \Phi < \varphi < 2\pi - \Phi,\ r > 0\}$.

We assume, that the wave field in Ω_o is $h + h_{inc} = h + \exp ik_o(-x\cos\vartheta_{inc} + y\sin\vartheta_{inc})$, (where $k_o = \frac{\omega}{c_o}$, ω – is the frequency, c_o is the velocity, h_{inc} is the incident wave) $h \in C^2_{loc}(\Omega_o)$ and satisfies Helmholtz equation in Ω_o:

$$(\Delta + k_o^2)(h + h_{inc}) = (\Delta + k_o^2)h = 0,\ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.1}$$

and the function $v \in C^2_{loc}(\Omega_i)$ and satisfies Helmholtz equation in Ω_i:

$$(\Delta + k_i^2)v = 0. \tag{1.2}$$

The next assumptions are: the functions h and v have continuous derivatives of the first order in the wedges without vertex: respectively $\Phi \leqslant \varphi \leqslant 2\pi - \Phi,\ r > 0$ and $0 \leqslant \varphi \leqslant \Phi,\ r > 0$, and boundary conditions are satisfied:

$$\nu_o(h + h_{inc}) = \nu_i v$$

$$\eta_o \frac{\partial}{\partial n}(h + h_{inc}) = \eta_i \frac{\partial}{\partial n} v \tag{1.3}$$

on the interfaces $\varphi = 0$ $r > 0$ and $\varphi = \Phi$ $r > 0$. Here $\frac{\partial}{\partial n}$ is derivative in direction of exterior (with respect to Ω_o) normal, ν_j and η_j are positive constants. We assume that the Meixner condition:

$$\int\limits_{r<r_0} \left(\frac{\partial u}{\partial x} \frac{\partial \bar{u}}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \bar{u}}{\partial y} \right) dxdy < \infty, \tag{1.4}$$

is fulfilled, where $r = \sqrt{x^2 + y^2}, r_0 = \text{const} < \infty$,

$$u = \begin{cases} v & \text{for } (x,y) \in \Omega_o \\ h & \text{for } (x,y) \in \Omega_i \end{cases}. \tag{1.5}$$

An important assumption is radiation condition: let u_{geom} be the sum of the incident, reflected and refracted waves (Lebeau "H-hypothesis" of nondegeneracy (see [1] chapter 2) takes place)[1] and $u - u_{geom} =: u_{diff}$ then

$$\int\limits_{\Sigma_R} \left| \frac{\partial u_{diff}}{\partial r} - iku_{diff} \right|^2 d\Sigma_R \underset{R\to\infty}{\longrightarrow} 0, \tag{1.6}$$

where

$$k = \begin{cases} k_o & \text{in } \Omega_o \\ k_i & \text{in } \Omega_i \end{cases}$$

The main result of the paper is:

Theorem

the problem in this formulation has solution and this solution is unique.

2. The method of the proof of the theorem

We use the approach developed in the book [1] and in the papers [2-3] and connected very closely with potential theory. Thi methods allow to prove the existence of solution of the problem under consideration in the form of sum of single layer potentials. This solution satisfies all conditions described above.

The proof of uniqueness is equivalent to the proof that the solution of corresponding homogeneous problem (let it be U) is equal to zero. The ideas of proof of uniqueness is based on constructing (just as in the paper [2]) of the solution of the problem of scattering by the transparent wedge the wave created by the point source of oscillation – i.e. a version of Green function. Existence of the Green function permit (with the help of integral Green formula) to produce integral representation the function U in the terms of the boundary conditions. From this representation follows that $U = 0$.

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References

[1]The fronts of the incident and every reflected and refracted wave are not ortogonal to the wedges boundary.

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