Gaussian Beam Summation Analysis of Half Plane Diffraction: A Full 3D Formulation

M. Katsav and E. Heyman

School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel. heyman@eng.tau.ac.il

Abstract

A Gaussian beam summation (GBS) representation for a three-dimensional half-plane diffraction of a Gaussian beam (GB) that hits arbitrarily close to the edge is presented. The expansion involves an angular spectrum of GB’s in the plane normal to the edge and a discrete phase-space decomposition along the edge. The field is thus described as a sum of GB’s emerging from a discrete set of points along the edge, at a discrete set of directions in a polar coordinate system. Expressions for the excitation amplitudes of the diffracted beams are derived and validated numerically.

1. Introduction

Gaussian beam summation (GBS) formulations are an important tool in wave theory as they provide a framework for ray-based construction of spectrally uniform solutions in complex configurations. In these formulations, the field is expanded into a spectrum of collimated beam propagators that emanate from the source domain in all directions, and thereafter are tracked locally in the medium and summed up at the observation points. The main advantages of these formulations are the spectral localization and the uniformity due to the fact that beam fields are insensitive to ray transition region (see [1, 2] and a recent review in [3]).

In many applications (e.g., in indoor or urban propagation) the ambient environment involves localized scatterers such as edges or corners. Various solutions for beam scattering by edges have been derived (e.g., [4] and [5] for frequency and time domain solutions, respectively). Yet, in order to be used consistently in GBS schemes, the scattered fields need to be expressed as a sum of GBs that emerge from the edge in all directions, thereby describing the edge in terms of its Gaussian-beam-to-Gaussian-beam (GB2GB) scattering matrix.

This scenario has been addressed in [6] for 2D half plane diffraction. Here we extend this work to an arbitrary 3D beam configuration. Our expansion schemes is based on the hybrid formulation introduced in [7], involving an angular spectrum of GB’s in the plane normal to the edge and phase phase-space decomposition along the edge. We present only the basic structure of the solution and some representative numerical results: The full analysis is presented in [8].

2. Problem formulation

We consider the diffraction of a general astigmatic GB by a perfectly conducting half plane in a 3D coordinate frame \( r = (x, y, z) \). The half plane is located in the \( y = 0 \) plane at \( x \geq 0 \), such that the edge axis coincides with the \( z \) axis. Referring to Figure 1, the beam axis may be displaced from the edge, intersecting the \( y = 0 \) plane at \((x_i, z_i)\) where \( x_i \) may positive or negative. Furthermore, the position \( z_i \) along the edge may be arbitrary, since in general the lattice of the expansion beams is determined a priori, regardless of the point of incidence. Finally, the incident beam direction is described by the polar angle \((\phi_i, \theta_i)\). Assuming an harmonic time-dependence \( e^{-i\omega t} \), the most general expression for the incident GB in a homogeneous medium is

\[
B_i(r) = A_i \sqrt{\frac{\text{det}(\Gamma(\sigma))}{\text{det}(\Gamma(0))}} e^{ik\left[\sigma + \frac{1}{2} \eta \Gamma(\sigma) \eta^t\right]}, \quad \text{where} \quad \Gamma(\sigma) = [\Gamma(0)^{-1} + \sigma I]^{-1}, \quad k = \frac{\omega}{c}.
\]  

Here \((\sigma, \eta)\) are coordinates along and transverse to the beam axis, respectively, with \( \eta = (\eta_1, \eta_2)^t \) and the superscript \( t \) denotes the transposed of a matrix. The so called “complex curvature” matrix \( \Gamma(\sigma) \) is complex symmetric with \( \text{Im} \Gamma(\sigma) \) positive definite. Its real and imaginary parts describe, respectively, the generally astigmatic phase-front curvature and Gaussian amplitude. Choosing the beam axis such that \( \sigma = 0 \) defines the point where it intersects the \( y = 0 \) plane, the transversal phase-front curvature and amplitude at the point of incidence are described by \( \Gamma(0) \).
The scattered field is described conveniently in the cylindrical coordinate frame \( r = (\rho, \phi, z) \) by [9]
\[
\begin{align*}
  u_s(r) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty} \! d\zeta \int_{-\infty}^{\infty} \! d\xi \, e^{ik\zeta^2} \hat{B}_i(\zeta, \xi) \frac{i}{4\pi} \int_{C_+} \! d\alpha \, D(\alpha, \alpha') e^{ik\rho \cos(\alpha - \phi)}, \\
  \zeta = \sqrt{1 - \xi^2},
\end{align*}
\]
\( \hat{B}_i \) represents the plane waves spectrum of the incident field \( B_i \) in the \( y = 0 \) plane, with \( (\zeta, \xi) \) being the spectral coordinate along the \( x, z \) axes, respectively, and \( \zeta \) is the transversal spectral wavenumber in the \( x, y \) plane associated with \( \xi \). Thus, the outer \( d\xi \, d\zeta \) integral in (2) expresses the incident field as a spectrum of plane waves, while the inner \( \alpha \) integral expresses the scattered field as a spectrum of plane waves emerging from the edge in the azimuthal direction \( \alpha \) with \( D(\alpha, \alpha') = \varepsilon \sec \frac{1}{2}(\alpha + \alpha') - \sec \frac{1}{2}(\alpha - \alpha') \) being the spectral diffraction coefficient for an incident plane-wave from the azimuthal \( \alpha' \) direction, with \( \varepsilon = \pm 1 \) for Dirichlet or Neumann boundary conditions, respectively [10]. \( \alpha' = \cos^{-1}(-\xi/\zeta) \) defines the azimuthal spectral angle of incidence. The \( \alpha \)-integration contours \( C_{\pm} \) in (2) are then defined such that if \( \phi \in (0, \pi) \) (i.e., \( y > 0 \)) the integration contour \( C_+ \) goes from \( i\infty \) to \( \pi - i\infty \) passing above the pole \( \alpha_{p+} = \pi - \alpha' \), while for \( \phi \in (\pi, 2\pi) \) (i.e., \( y < 0 \)) the integration contour is \( C_- \) passing from \( \pi + i\infty \) to \( 2\pi - i\infty \) passing below the pole \( \alpha_{p-} = \alpha' - \pi \). The total field is expressed now by
\[
  u(r) = B_i(r) + u_s(r).
\]
Alternatively, one may extract the reflected field solution and express the total field as
\[
  u(r) = u_h(r) + \tilde{u}_s(r), \quad \text{where } u_h(r) = B_i(r) + B_r(r) \text{ for } y > 0 \text{ and } u_h(r) = 0 \text{ for } y < 0,
\]
with \( B_r(r) \) being the reflected beam field assuming an infinite reflecting plane at \( y = 0 \). The “modified” scattered field \( \tilde{u}_s(r) \) in (4) is given by the same spectral integrals as in (2) except that the integration contours \( C_{\pm} \) pass now on the other side of the poles.

Both formulations in (3) and (4) are exact and applies for all incident beam directions. The former, however, is more “natural” when the incident beam does not intersect the half plane, i.e., when \( x_1 < 0 \) and in particular when \( x_1 \ll -W_i \) where \( W_i \) is the beamwidth of \( B_i \). In this case the field is dominated by \( B_i \) and \( u_s \) represents a relatively weak perturbation due to the half plane. Likewise, the formulation in (4) is more “natural” when \( x_1 > 0 \) where the “modified” scattered field \( \tilde{u}_s(r) \) expresses the relatively weak perturbation to the reflected field \( B_r \) due to the half plane.

### 3. A GBS expansion

Referring to Figure 1, the beam expansion of the scattered field involves a phase-space beam expansion along the edge, and an azimuthal spectrum of beams around the edge. This scheme has been introduced in [7] in the context of the simpler problem of radiation from a line-source distribution. The derivation of this hybrid combination involves two phases: In the first, the \( \alpha \) integration in (2) is discretized to derive an azimuthal GBS expansion in a plane perpendicular to the \( z \)-axis, while in the second, the \( \zeta \) integration is expanded using a windowed Fourier transform (WFT) frame expansion [11,7] to obtain the GBS expansion along the \( z \)-axis. The final result for the scattered field \( u_s \) is
\[
  u_s(r) = \sum_{mnj} a_{mnj} B_{mnj}(r), \\
  B_{mnj}(r) \approx \sqrt{-iF_1} \sqrt{-iF_2} e^{jk\left(\sigma + \frac{1}{2} + \frac{\eta_1^2}{\sigma^2} + \frac{\eta_2^2}{\sigma^2}\right)}.
\]
This expression represents the field as a sum of GB’s \( B_{mnj}(r) \) weighted by the expansion coefficients \( a_{mnj} \), emerging from the points \( z_m \) on the in the the direction \( (\theta_m, \alpha_j) \) (Figure 1). The lattice of initiation points \( z_m \) and the polar angles \( \theta_m \) is determined via a Gabor-type phase-space expansion along the edge [11,7], while the azimuthal angles \( \alpha_j \) are determined by discretization of the \( \alpha \) integral [7]. In the expression for \( B_{mnj} \), \( \sigma \) is a coordinate along the beam axis from the initiation point \( z_m \) and \( (\eta_1, \eta_2) \) are the coordinates transverse to this axis in the \( (\theta, \phi) \) directions (Fig. 1). This expression has the standard form of an astigmatic GB with \( (\eta_1, \eta_2) \) being the principle axes, having a waist at \( \sigma = 0 \) and collimation lengths \( F_{1,2} \). These parameters are determined by the window functions that are used in the expansion [7]. The expansion coefficients \( a_{mnj} \) depend on the spectrum of the incident beam along \( z \) and transverse to the \( z \) axis, and on the integrated diffraction coefficients in the \( \alpha_j \) direction due to the plane-wave spectrum of the incident beam field. The expressions for these coefficients are therefore quite complicated, since the integral contains also spectral poles of \( D \). As expected, \( a_{mnj} \) are strongest for \( \alpha_j \) near the incident and reflected beam directions. Further details and analytic expressions are given in [8].
The GBS expansion in (5) is written for the scattered field $u_s$, but a similar expression is obtained for the modified scattered field $\tilde{u}_s$. The only difference is the use of somewhat modified expansion coefficients $\tilde{a}_{mnj}$. As discussed after (4), both formulations may be applied for all incident beam directions. Yet the $u_s$ formulation is more “natural” and numerically more accurate when the incident beam does not intersect the half plane, i.e., when $x_i < 0$, in which case the field is dominated by $B_i$ and $u_s$ represents a only a weak perturbation due to the edge. Likewise, the $\tilde{u}_s$ formulation is more “natural” and more accurate numerically when $x_i > 0$, in which case the field is dominated by the reflected beam $B_r$ so that $\tilde{u}_s(r)$ is the relatively weak perturbation to edge.

4. Numerical example

Referring to Fig. 1, we assume that the incident GB (1) is stigmatic (circular symmetric) with $\Gamma_i = ib^{-1}I$, with $b = 500$ representing the collimation distance of the beam, and the beamwidth is given by $W_i = \sqrt{b/k}$. The units are chosen such that $k = 1$. The incident beam direction is taken to be $(\theta_i, \phi_i) = (\pi/3, \pi/4)$ and the beam axis intersects the half plane at $x_i = 2$ and $z_i = 0$. Following the discussion above, the “natural” representation in this case is the $u_s$ formulation. The considerations for choosing the parameters and the lattice of the expansion beams are quite complicated and are not considered here. We only comment that the beams emerge from $z_m = 137.3 m$ in the directions $\theta_n = \cos^{-1} \zeta_n$ with $\zeta_n = 0.0114 n$. Further details are given in [8]

Fig. 2 depicts a map of the coefficients $a_{mnj}$ in the $(z_m, \zeta_n)$ phase space. The $\alpha_j$ beam direction is taken to be at the reflection boundary $3\pi/4$. One readily observe that the strongest excitation is of the beams emerging from $z_m = 0$ in the $\theta_n = \pi/3$ direction. Finally, Fig. 3 depicts the field at a distance $\rho = 3500$ from the edge, as a function of the observation coordinates $\phi$ and $z$. Fig. 3(a) depicts the modified scattered field $\tilde{u}_s$. As a reference we show in Fig. 3(b) the geometrical optics (GO) field i.e., the reflected field $B_r$ for $y > 0$ assuming an infinite reflecting surface at $y = 0$ and the incident field $B_i$ for $y < 0$ assuming no scatterer at the $y = 0$ plane. The normalized error distribution $|u^{\text{GO}}_s - u^{\text{ref}}|/\max |u^{\text{ref}}|$ of the GBS calculation relative to the exact reference solution in % is depicted in Fig. 3(c).

5. Concluding remarks

We presented a hybrid GBS scheme for diffraction an incident generally astigmatic GB by a half plane, involving an angular spectrum of GB’s in the plane normal to the axis and phase phase-space decomposition along the edge. The scheme has been validated and calibrated using a numerical example.

6. Acknowledgement

This work is supported in part by the Israeli Science Foundation, under Grant No. 674/07, and by NATO’s Public Diplomacy Division in the framework of “Science for Peace” program, under Grant No. SfP982376.

7. References


Figure 1: Physical configuration: A general GB (1), marked by a heavy arrow, impinges on a half plane (wedge angle $\Psi = 2\pi$) at direction ($\phi_i$, $\theta_i$). It intersects the $y = 0$ plane at an arbitrary point $x_i = (x_i, z_i)$, where $x_i$ can be positive or negative. The beam expansion of the scattered field (dashed arrows) involves a phase-space beam expansion along the edge, and an azimuthal spectrum of beams around the edge. The beams emerge from the points $z_m, m = 0, \pm 1, \ldots$, along the edge with conical angle $\theta_{nj} = \cos^{-1} \zeta_n, n = 0, \pm 1, \ldots$, and azimuthal angles $\alpha_j, j = 0, 1, \ldots$.

Figure 2: A phase-space distribution of the expansion coefficients $\hat{a}_{mnj}$ in the ($z_m, \zeta_n$) plane.

Figure 3(a): The absolute value of the modified scattered field $\hat{u}_s$ calculated numerically via the GBS.

Figure 3(b): The GO field, shown as a reference for the solution in Fig 3(a).

Figure 3c: The normalized error in $\%$ of the GBS calculation relative to an exact reference solution.