Uniform asymptotic solution in the problem of acoustic plane wave diffraction by an impedance circular cone

Mikhail A. Lyalinov [1], ***Ning Yan Zhu*** [2], ***Valery E. Grikurov*** [3]

[1]Dept. of Mathematical Physics, Institute of Physics, Saint-Petersburg University, Ulianovskaya 1-1, Petershof, 198504, Russia, lyalinov@yandex.ru

[2]Institut für Hochfrequenztechnik, Universität Stuttgart, D-70569 Stuttgart, Germany, zhu@ihf.uni-stuttgart.de

[3]Dept. of Mathematical Physics, Institute of Physics, Saint-Petersburg University, Ulianovskaya 1-1, Petershof, 198504, Russia, grikurov@mph.phys.spbu.ru

Abstract

The main goal of this work is to develop a uniform asymptotic expression for the scattered field which is valid for all directions of observation outside the conical surface. The parabolic-cylinder (PC) functions ansatz is exploited for the derivation. We propose a procedure of derivation of the coefficients in the PC ansatz in the leading approximation for an impedance circular cone.

1 Introduction

The incident acoustic plane wave completely illuminates **a semi-infinite** right-circular conical surface with impedance boundary conditions. The scattered field satisfies the Helmholtz equation in the exterior of the conical surface, radiation conditions at infinity and the Meixner's condition at the vertex of the cone. The main goal of this work is to develop a uniform asymptotic expression for the scattered field which is valid for all directions of observation. The parabolic-cylinder (PC) functions ansatz is exploited for derivation which is known from the work by A.L. Brodskaya, A.V. Popov, S.A. Hoziossky (1976) and further developed by V.M. Babich (2004, also for the em waves). We propose a procedure of derivation of the coefficients in the PC ansatz in the leading approximation for an impedance convex cone ($\eta = \sin\zeta$ is the surface impedance, $\Re\,\eta > 0$).

It is well known that the rays reflected from the conical surface are terminated by a surface of singular directions which correspond to the directions in which the diffraction coefficient of the spherical wave from the vertex has a singularity so that the non-uniform expression for it is not applicable. The uniform expression based on the PC ansatz is valid across the surface of singular directions and exploits special ray coordinates for the reflected and scattered spherical waves. In order to specify the coefficients of the mentioned asymptotic ansatz we use an analytical-numerical procedure of derivation of the non-uniform diffraction coefficients recently developed by Lyalinov and Zhu (2007) both in the domains illuminated and not illuminated by the reflected rays and having the mentioned singularities. In order to determine the unknown coefficients in the uniform expressions we apply a traditional matching procedure. Namely, we take into account the asymptotics of PC functions and find the sought-for coefficients by matching them with the non-uniform (but known) expressions for the diffractions coefficients. In the matching procedure analysis of the singularity of the diffraction coefficient plays an important role.

2 Ansatz

The whole field outside an impedance cone S completely illuminated by a plane incident wave $U^i(kr,\omega,\omega_0)$ is given by the sum

$$\hat{U}(kr,\omega,\omega_0) = U^i(kr,\omega,\omega_0) + U(kr,\omega,\omega_0), \quad k = \Omega/c, \tag{1}$$

where

$$U^i(kr,\omega,\omega_0) = e^{-ikr\cos\hat{\theta}(\omega,\omega_0)} \tag{2}$$

is the incident wave, $\omega_0 = (\theta_0, \varphi_0)$ is the unit vector attached to the direction of incidence, $\omega = x/r = (\theta, \varphi)$, $r = |x|$, $\cos\hat{\theta}(\omega, \omega_0) = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0)$, c is the wave speed in the acoustic medium,

$$U(kr, \omega, \omega_0) = W(kr, \omega, \omega_0) + U_{sw}(kr, \omega, \omega_0)\,, \tag{3}$$

U_{sw} is the surface wave, $x = (r, \theta, \varphi)$,

$$\begin{aligned} W(kr, \omega, \omega_0) = {} & \frac{\exp[ikl(x)]}{k^{1/4}} \left[D_{-3/2}(\sqrt{k}e^{-i\pi/4}m(x)) \left\{ A_0(x) + \frac{A_1(x)}{ik} + \dots \right\} + \right. \\ & \left. + \frac{e^{i\pi/4}}{\sqrt{k}} D'_{-3/2}(\sqrt{k}e^{-i\pi/4}m(x)) \left\{ B_0(x) + \frac{B_1(x)}{ik} + \dots \right\} \right] + \frac{e^{ikr}}{k} \left\{ C_0(x) + \frac{C_1(x)}{ik} + \dots \right\}, \end{aligned} \tag{4}$$

with $(k \gg 1)$

$$l(x) = r\sin^2\hat{\theta}'(\omega, \omega_0)/2, \;\; m(x) = -2\sqrt{r}\cos\hat{\theta}'(\omega, \omega_0)/2$$

and $\hat{\theta}'(\omega, \omega_0)$ is the length of the broken geodesics on the unit sphere (see e.g. Babich et al., 2000), corresponding to the reflected wave, $\hat{\theta}'(\omega, \omega_0) = \pi$ describes the conical surface, terminating the rays reflected from the cone. It is the surface of singular direction, where the diffraction coefficient of the spherical wave from the vertex has a singularity of $O\left([\cos\hat{\theta}'/2]^{-3/2}\right)$ as $\cos\hat{\theta}'/2 \to 0$. In the domain $\hat{\theta}'(\omega, \omega_0) > \pi$ only the spherical wave from the vertex exists in the far scattered field. Such domain is called oasis. The domain $\hat{\theta}'(\omega|_S, \omega_0) \le \hat{\theta}'(\omega, \omega_0) < \pi$ is illuminated by the reflected rays. The coefficients A_0, B_0, C_0, which are of primary importance, are smooth across the singular directions. After substituting the ansatz (4) into the Helmholtz equation and equating the terms of the same powers of k, we obtain the transport equations for the determination of A_0, B_0, C_0 and solving them

$$\begin{aligned} A_0(x) &= \frac{d_+(\omega, \omega_0)}{r} \frac{m^{3/2}(x)}{2} + d_-(\omega|_\sigma, \omega_0) \sqrt{\frac{\sin\hat{\theta}'(\omega, \omega_0)}{\sin\hat{\theta}'(\omega|_\sigma, \omega_0)} \frac{J(\omega|_\sigma, \omega_0)}{J(\omega, \omega_0)}} \frac{m^{-1/2}(x)}{2}\,, \\ B_0(x) &= -\frac{d_+(\omega, \omega_0)}{r} m^{1/2}(x) + d_-(\omega|_\sigma, \omega_0) \sqrt{\frac{\sin\hat{\theta}'(\omega, \omega_0)}{\sin\hat{\theta}'(\omega|_\sigma, \omega_0)} \frac{J(\omega|_\sigma, \omega_0)}{J(\omega, \omega_0)}} m^{-3/2}(x)\,, \\ C_0(x) &= \frac{C(\omega, \omega_0)}{r}\,, \end{aligned} \tag{5}$$

where σ is the curve of intersection of the conical surface and the unit sphere centered at the vertex with a hole Σ, $(\sigma = \partial\Sigma)$ cut by the conical surface, J is the spreading of the broken (reflected from σ) geodesics of the sphere. The coefficients $d_\pm$, C are still unknown.

3 Matching with the non-uniform asymptotics

Instead of $d_\pm$ we equivalently determine the values

$$A(\omega, \omega_0) = r^{1/4}\, A_0(x), \quad B(\omega, \omega_0) = r^{3/4}\, B_0(x) \tag{6}$$

exploiting matching of the uniform expression (4) with the local one in the domain illuminated by the reflected rays

$$\begin{aligned} W(kr, \omega, \omega_0) = {} & e^{-ikr\cos\hat{\theta}'(\omega, \omega_0)} R_\zeta \sqrt{\frac{\sin\hat{\theta}'(\omega, \omega_0)}{\sin\hat{\theta}'(\omega|_\sigma, \omega_0)} \frac{J(\omega|_\sigma, \omega_0)}{J(\omega, \omega_0)}} \left(1 + O\left(\frac{1}{kr\cos^2\hat{\theta}'/2}\right)\right) + \\ & + \frac{e^{ikr}}{-ikr} D(\omega, \omega_0) \left(1 + O\left(\frac{1}{kr}\right)\right)\,, \end{aligned} \tag{7}$$

where R_ζ is the reflection coefficient and $D(\omega, \omega_0)$ means the diffraction coefficient.

The spectral function $u_\nu(\omega,\omega_0)$ solves the known boundary-value problem on the unit sphere (see (13) below or, e.g., [Lyalinov; 2003]). Taking into account the leading terms of the asymptotics of the PC function

$$D_{-3/2}(z) \sim e^{-z^2/4} z^{-3/2} - \frac{\sqrt{2\pi}}{\Gamma(3/2)} e^{-i3\pi/2} e^{z^2/4} z^{1/2}\,, \;\; \arg z = 3\pi/4,\;\; |z| \to \infty,$$

we obtain $(m(x) \to -\infty)$

$$\begin{aligned} &4e^{-i\pi/8}\, A(\omega,\omega_0)[\cos\hat\theta'/2]^{1/2} + 4e^{-i9\pi/8}\, B(\omega,\omega_0)[\cos\hat\theta'/2]^{3/2} = R_\zeta \sqrt{\frac{\sin\hat\theta'(\omega,\omega_0)}{\sin\hat\theta'(\omega|_\sigma,\omega_0)}\,\frac{J(\omega|_\sigma,\omega_0)}{J(\omega,\omega_0)}}, \\ &2^{-3/2}e^{-i9\pi/8}\, A(\omega,\omega_0)[\cos\hat\theta'/2]^{-3/2} + 2^{-3/2}e^{-i9\pi/8}\, B(\omega,\omega_0)[\cos\hat\theta'/2]^{-1/2} + C(\omega,\omega_0) = iD(\omega,\omega_0). \end{aligned} \tag{8}$$

In (8) we have two equations for three unknown functions A, B, C. In order to get the third equation we should analyse the behaviour of the diffraction coefficient $D(\omega,\omega_0)$ near the singular directions ω_s, $\hat\theta'(\omega_s,\omega_0) = \pi$. We have the representation near the singular directions $(\omega \sim \omega_s)$

$$iD(\omega,\omega_0) = \frac{iD_1(\omega,\omega_0)}{[\cos\hat\theta'(\omega,\omega_0)/2]^{3/2}} + iD_2(\omega,\omega_0), \tag{9}$$

where $D_{1,2}(\omega,\omega_0)$ are regular functions near the singular directions ω_s. Provided one could compute the value $iD_2(\omega,\omega_0)$, from the formulae (8), (9) one has

$$C(\omega,\omega_0) = iD_2(\omega,\omega_0), \tag{10}$$

which gives the third desired equation.

4 Derivation of the regular part $D_2(\omega,\omega_0)$ of the diffraction coefficient

The expression for the regular part of the diffraction coefficient can be given in the form

$$iD_2(\omega,\omega_0) \;=\; \frac{2}{i}\int\limits_{-i\infty}^{i\infty} \nu\, e^{i\pi\nu}\left[u_\nu(\omega,\omega_0) - (1-\chi_a(\nu))u^r_\nu(\omega,\omega_0)\right] d\nu\,, \tag{11}$$

where $\chi_a(\nu)$ is a cut-off function which is zero outside the interval $i\nu \in [-a-1, a+1]$ and is unit as $i\nu \in [-a, a]$ for some large $a > 0$. In the integrand the ray series $(\nu \to \pm i\infty)$ in

$$u_\nu(\omega,\omega_0) = u^r_\nu(\omega,\omega_0) + O(\nu^{-1}e^{\pm i\nu(\pi+\delta)}), \quad u^r_\nu(\omega,\omega_0) \sim \sum_{j=0}^{\infty} \frac{e^{\pm i\sqrt{\nu^2-1/4}\,\hat\theta'(\omega,\omega_0)}\, b^j_\pm(\omega,\omega_0)}{\left(\sqrt{\nu^2-1/4}\right)^{j+1/2}} \tag{12}$$

are subtracted from the spectral function ensuring the estimate for the integrand $O(e^{-|\nu|\delta})$ with $\delta = \Re\,\eta > 0$. We imply that $u^r_\nu(\omega,\omega_0)$ can be computed explicitly as a formal asymptotic solution $(\nu \to \pm i\infty)$ of the spectral problem on the unit sphere with a hole Σ, $(\sigma = \partial\Sigma)$

$$\left(\triangle_\omega \,+\, (\nu^2 - 1/4)\right)\, u_\nu(\omega,\omega_0) \,=\, 0, \qquad \left.\frac{\partial \hat U_{\nu+1}}{\partial\theta}\right|_\sigma \,-\, \left.\frac{\partial \hat U_{\nu-1}}{\partial\theta}\right|_\sigma \,=\, (-2\nu)\eta\, \hat U_\nu\Big|_\sigma\,, \tag{13}$$

$\hat U_\nu \,=\, u_\nu + u^i_\nu$, $\triangle_\omega$ is the Laplace-Beltrami operator on S^2, $u^i_\nu(\omega,\omega_0) \;=\; -\dfrac{P_{\nu-1/2}\left(-\cos\hat\theta(\omega,\omega_0)\right)}{4\cos(\pi\nu)}$, $P_{\nu-1/2}(x)$ is the Legendre function. It is important to mention that the integral (11), however, with the integrand $\nu\, e^{i\pi\nu}\, u^r_\nu(\omega,\omega_0)$ exactly gives the singular part of the diffraction coefficient.

5 Conclusion

In the present report we propose a theoretical procedure to derive a uniform expression for the far field. However, its numerical implementation is not yet completed though we expect that after an appropriate adaptation and, perhaps, modifications it will give satisfactory numerical results. The latter depend on efficiency of calculations of the non-uniform scattering diagram and its regular part.

Two authors (MAL, VEG) were supported in part by the NATO SP Grant - 982376.

6 References

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