Performance of Conformal PML for the Mixed Finite-Element Time-Domain Method

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Abstract

We provide a performance analysis for the conformal perfectly matched layer (PML) applied to the mixed finite-difference time-domain (FETD) method. The mixed FETD method is based on the first order coupled Maxwell's equations and include both electric field intensity and magnetic flux density as unknowns. As opposed to finite-element method based on the second-order wave equation, the constitutive relation and curl equations are decoupled in mixed finite-elements and they can be treated separately. This allows straightforward implementation for the anisotropic and dispersive tensors required for conformal PML. A conformal PML implementation is particularly important for scattering problems because it can reduce the buffer space significantly when compared to rectangular PML without any computational drawbacks. We study the performance of conformal PML for mixed FETD with respect to several PML parameters such as number of layers and conductivity profile.

1. Introduction

The finite-element time-domain (FETD) method has been widely used in solution of Maxwell's equations on structured meshes. FETD has been traditionally based upon the second-order wave equation for either electric or magnetic field. This allows for the expansion of the unknown field by a single type of basis function [1]. It has been shown that an *edge* element expansion conforms to a discrete version of the de Rham complex [2] and provides the correct physical representation for the fields. However this condition is enforced in a frequency-dependent fashion and it is weakened as the frequency approaches to zero. In time-domain, a similar problem is observed in the form of spurious linear growth of gradient-like fields (spurious solutions of the wave equation).

A mixed vector FETD has been recently introduced as an alternative to the second-order wave equation FETD [3-6]. The mixed FETD method is based upon Maxwell's first-order curl equations and uses both the electric and magnetic fields as unknowns. Edge elements (Whitney 1-form) are used to represent the electric field intensity, E, and face elements (Whitney 2-form) are used to represent the magnetic flux density, B. The update equations resemble that of finite-difference time-domain with the exception that the mixed finite-element update necessitates a sparse matrix solver. Another important similarity is in the implementation of constitutive relations: unlike traditional FETD, they appear decoupled from the curl equations in mixed FETD; this allows a more straightforward implementation of complex media [5, 6].

A conformal perfectly matched layer (PML) has been recently introduced as an absorbing boundary condition for FETD [6]. The conformal PML can be represented by anisotropic material tensors that absorb incoming waves without producing reflections. Hence, they can be used to terminate boundaries of open domain problems [7, 8]. A conformal PML can be used for any problem with a convex boundary and may lead to significant computational savings when compared to a (traditional) rectangular PML implementation, especially in scattering problems involving impenetrable scatterers [9]. As noted above, implementation of conformal PML in the mixed FETD method is straightforward since the constitutive equations are decoupled from the curl equations. We provide here a performance analysis for the conformal PML for the mixed FETD with respect to various parameters such as number of layers and PML conductivity profile.

2. Conformal PML for the Mixed *E-B* FETD

The mixed *E-B* FETD method is based on an edge element (Whitney 1-form), W_i^1, expansion for the electric field E, and a face element (Whitney 2-form), W_i^2 expansion for the magnetic flux B [2]. The semi-discrete Maxwell's equations for mixed vector *E-B* FETD can be written as [6, 10]:

$$[*_{\varepsilon}]\frac{\partial}{\partial t}\mathbf{E} = \left[D_{curl}^{*}\right]\left[*_{\mu^{-1}}\right]\mathbf{B} \tag{1}$$

$$\frac{\partial}{\partial t}\mathbf{B} = -\left[D_{curl}\right]\mathbf{E} \tag{2}$$

where $\mathbf{E}=[e_1,e_2,\ldots,e_{Ne}]^T$, $\mathbf{B}=[b_1,b_2,\ldots,b_{Nf}]^T$ are electric field intensity and magnetic flux density unknown vectors, $[D_{curl}]=[D^*_{curl}]^T$ are the incidence matrices on primal and dual grids of the finite-element mesh, and $[*_\varepsilon]$ and $[*_{\mu-1}]$ are the Hodge matrices (generalized mass matrices) for permittivity and inverse permeability, respectively. If $\varepsilon(r)$ and $\mu(r)$ denote the permittivity and permeability at location r, and representing the computational volume as Ω, the Hodge matrices $[*_\varepsilon]$ and $[*_{\mu-1}]$ are given by [4, 6, 10]

$$[*_\varepsilon]_{ij} = \int_\Omega W_i^1 \cdot \varepsilon(r) \cdot W_j^1 d\Omega \tag{4}$$

$$\left[*_{\mu^{-1}}\right]_{ij} = \int_\Omega W_i^2 \cdot \mu^{-1}(r) \cdot W_j^2 d\Omega \tag{5}$$

These matrices are sparse and positive-definite. The permittivity and permeability tensors for the conformal PML is represented in a local orthogonal Darboux frame [7] with unit vectors u_ρ, u_φ, u_z, along the respective coordinates (ρ, φ, z) in a 2D TE_z problem as

$$\begin{aligned} \varepsilon(r) &= \varepsilon(\rho) = u_\rho u_\rho \frac{\gamma(\rho)}{s(\rho)} + u_\varphi u_\varphi \frac{s(\rho)}{\gamma(\rho)} \\ \mu(r) &= \mu(\rho) = u_z u_z \gamma(\rho) s(\rho) \end{aligned} \tag{6}$$

with

$$\gamma(\rho) = \frac{\rho_0}{\rho} + \frac{1}{\rho}\int_{\rho_0}^{\rho_0+l} s(\rho) d\rho \qquad\qquad s(\rho) = 1 + \frac{\sigma_\rho(\rho)}{j\omega\varepsilon_0} \tag{7}$$

Here, $s(\rho)$ and $\gamma(\rho)$ are PML stretching variables along the radial and angular directions in the local coordinate system, respectively. Both these variables depend on the PML conductivity parameter σ_ρ. The conformal PML interface is introduced at ρ_0, where $\rho=\rho_0+l$, ρ_0 is the local radius of curvature, and l is the distance from the interface into the PML, see more details in [6]. A face based definition is used for the material parameters such as $\sigma_\rho(k)$, $s(k)$, $\gamma(k)$, $\varepsilon(k)$ and $\mu(k)$, where k is the face index. The update equations are obtained by substituting (6) into, (4) and (5) [6].

3. Performance Analysis

The performance of the conformal PML for mixed *E-B* FETD method with respect to PML parameters is analyzed in a 2D TE_z problem. Unless stated otherwise, the following parameters are used in the numerical tests. A magnetic current line source with an ultra-wideband Blackman-Harris pulse driving function is used as the excitation. The central wavelength of the pulse is chosen as λ_0=0.6 m. A sparse Cholesky factorization with a drop tolerance value 10^{-10} is used to solve the sparse matrix equation. The time-step is chosen with respect to the shortest edge length in the mesh l_{min} as $\Delta t=c_N l_{min}/v_c$., where c_N=0.2 is the Courant number.

The finite-element mesh used in the simulations is depicted in Fig. 1. A mesh generation algorithm with a maximum area constraint $\Omega_k^2<3l_e^4/16$ is used where l_e=2 cm is the mesh resolution in terms of typical edge-length and Ω_k is the area of the kth face. The edge length in the radial direction in conformal PML is chosen as $l_p=0.7l_e$. The reflection error is calculated at two different locations indicated in the figure: A and B, which are in the vicinity of a PML interface section with small and large radii of curvature, respectively. At each location, a source-receiver pair is placed 6 cm away from the PML interface and 4 cm away from each other. For the reference result, a FETD simulation with the same mesh extruded by an appropriate number of free-space layers is used. The extruded part of the mesh consists of convex quadrilateral tiles composed of two triangular faces each similar to the PML region. A separate simulation with a mesh terminated by perfect electric-conductor (PEC) is used for normalization purposes. If we denote the magnetic fields at the probe location in simulations with the PML mesh, free-space extruded mesh, and the PEC-terminated mesh as h_{pml}, h_{fs} and h_{pec} respectively, and n is the time step index, the reflection error is calculated by $\Gamma(n)=20\log_{10}(|h_{pml}(n)-h_{fs}(n)|/\max_n|h_{pec}(n)-h_{fs}(n)|)$.

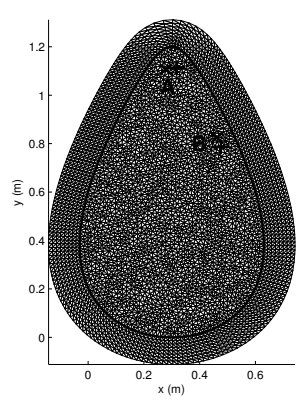


Fig. 1 FEM mesh used in the analysis. The source/receiver (+) / (×) pair is at locations A and B, as indicated.

A polynomial profile with maximum conductivity σ_{max} and polynomial coefficient M is used in the PML region. The profile is given as $\sigma_\rho(k) = \sigma_{max}(l/d)^M$, where d is the total width of PML and l is the distance between the layer associated with face k and the PML interface.

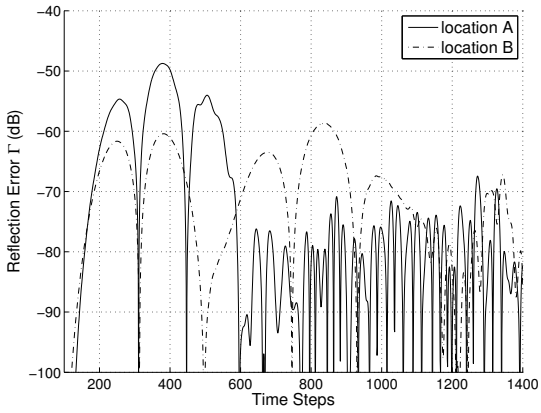
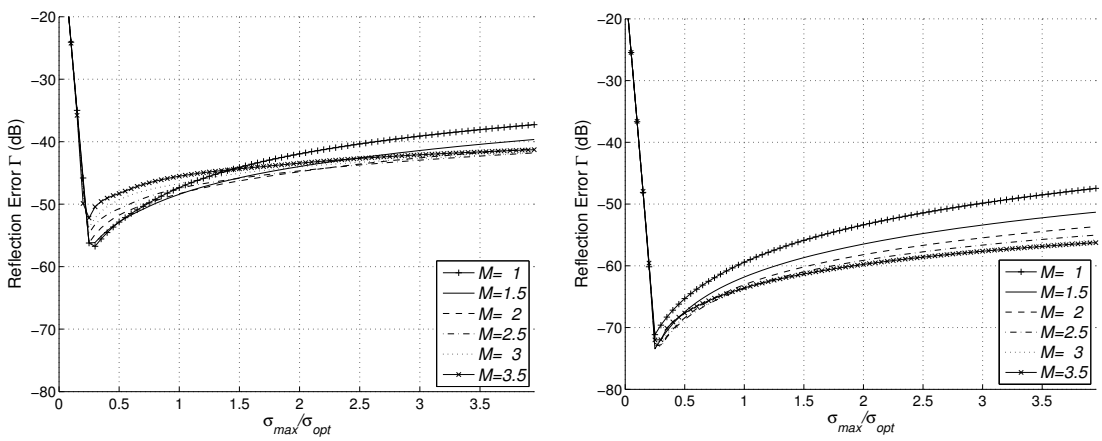


Fig. 2 Reflection Error in time at location A and B (N_{PML}=12, σ_{max}=0.5σ_{opt}, M=2).

The reflection error Γ(n) in time at both location A and B for N_{PML}=12, σ_{max}=0.5σ_{opt} and M=2 is shown in Fig. 2**.** The maximum error observed at location A is found to be approximately 10 dB higher than the error at location B, mainly due to the difference in the radius of curvature associated with these points. For instance, it can be seen from Fig. 1 that, location A is in the vicinity of PML with small radius of curvature and larger faces, which increases the numerical dispersion at this point. It can also be seen from the figure that the error levels are lower than a finite-difference time-domain (FDTD) simulation with similar parameters. One possible explanation is that the face-based material definition in the finite-element method assigns only one material step at each PML layer, as opposed to two in (staggered grid) FDTD.

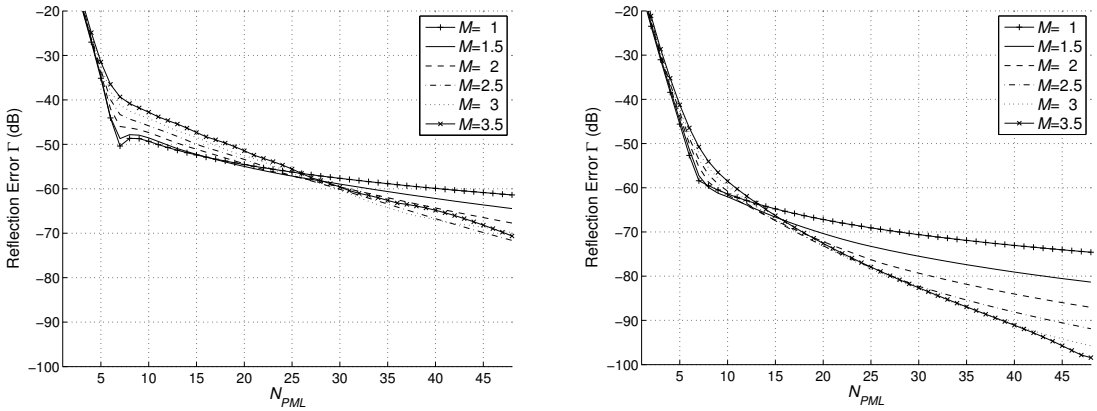
Fig. 3 shows the reflection error at both locations for N_{PML}=16 with respect to normalized maximum conductivity ($\sigma_{max}/\sigma_{opt}$), where $\sigma_{opt}=(M+1)/(150\pi\sqrt{\varepsilon_0}\, l_p)$ is the optimum value used for FDTD, and the PML grading polynomial coefficient M. It can be seen from the figure that the reflection error at location A is 15 dB higher than that at B due to lower radius of curvature. The optimum value for maximum conductivity is observed to be approximately half of σ_{opt}. Again, this may be due to the face-based material definition in the finite-element method which assigns a single material step per layer, as opposed to one in FDTD. This increases the error due to conductivity steps and requires a slightly smoother profile for a given N_{PML} or M.



(a) location A (b) location B

Fig. 3 Reflection error at locations A and B with respect to normalized maximum conductivity and PML grading polynomial coefficient (N_{PML}=16). Optimum maximum conductivity is observed to be half of FDTD optimum value, σ_{opt}.

Fig. 4 shows the reflection error at both locations with respect to number of PML and several different M. The error follows the theoretical reflection values of PML and converges rapidly for N_{PML}<7. The speed of convergence is reduced for larger number of layers, where it is dominated by a separate error mechanism possibly related to the numerical dispersion error associated with the higher frequency range of the source spectrum.



(a) location A (b) location B

Fig. 4 Reflection error at locations A and B with respect to number of layers and the PML grading polynomial coefficient ($\sigma_{max}/\sigma_{opt}$=0.5).

4. Conclusion

The performance of conformal PML for the *E-B* mixed vector FETD method has been analyzed. The mixed finite-element method is based on first-order coupled Maxwell's equations and provides a natural path to implement complex media, such as conformal PML. Conformal PML reduces the buffer zone and may lead to significant computational savings. The optimum maximum conductivity for the mixed FETD is found to be about half of that in FDTD simulations under similar conditions.

5. References

1. J-F Lee, R. Lee, and A. Cangellaris, "Time-domain finite-element methods," *IEEE Trans. Antennas Propagat.*, vol. 45, no. 3, 1997, pp. 430-442.

2. A. Bossavit, *Computational Electromagnetics: Variational Formulations, Complementarity, Edge Currents*, Academic Press, San Diego, 1998.

3. J. Koning, R. N. Rieben, and G. H. Rodrigue, "*Vector finite-element modeling of the full-wave Maxwell equations to evaluate power loss in bent optical fibers*," IEEE J. Lightwave Technol., vol. 23, no. 12, Dec. 2005, pp. 4147-4154.

4. B. He and F. L. Teixeira, "*A sparse and explicit FETD via approximate inverse Hodge (mass) matrix,*" IEEE Microw. Wireless Components Lett., vol. 16, no. 6, Jun. 2006, pp. 348-350.

5. B. Donderici, F. L. Teixeira, "*Mixed finite-element time-domain method for transient Maxwell equations in doubly-dispersive media*", IEEE Microw. Theory and Tech., vol. 56, no 1, Jan. 2008, pp. 113-120.

6. B. Donderici, F. L. Teixeira, "*Conformal perfectly matched layer for the mixed finite element time-domain method*", IEEE Trans. On Anten. and Propagat., 2008, in press.

7. F. L. Teixeira and W. C. Chew, "*Systematic derivation of anisotropic PML absorbing media in cylindrical and spherical coordinates,*" IEEE Microw. Guided Wave Lett., vol. 7, no. 11, 1997, pp. 371-373.

8. F. Collino and P. Monk, "*The perfectly matched layer in curvilinear coordinates,*" SIAM J. Sci. Computing, vol. 19, 1998, pp. 2061-2090.

9. F. L. Teixeira and W. C. Chew, "*Analytical derivation of a conformal perfectly matched absorber for electromagnetic waves,*" Microw. Opt. Tech. Lett., vol. 17, no. 4, 1998, pp. 231-236.

10. B. He and F. L. Teixeira, "*Mixed E-B finite elements for solving 1-D, 2-D, and 3-D time-harmonic Maxwell curl equations*," IEEE Microw. Wireless Comp. Lett., vol. 17, no. 5, 2007, pp. 313-315.