Information content of the radiated field over two bounded domains in the Fresnel zone

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Abstract

We consider the problem of determining the information content of the field radiated by a bounded monochromatic rectilinear electric source observed over two bounded rectilinear domains parallel to the source and located in its Fresnel zone. The problem is tackled by means of the Singular Value Decomposition of the relevant linear operator so that the information content is defined as the number of "significant" singular values. In particular, by adopting an approximate mathematical model of the problem, the singular value behavior is very well predicted and the role of the geometrical parameters highlighted.

1. Introduction

Determining the information content of the radiated field, also addressed in the literature as the number of degrees of freedom (NDF) or even the essential dimension of the radiated field, is a relevant problem in optics, electromagnetic and acoustics [1]. The information content represents the number of significant and independent parameters needed to represent the radiated field and in inverse problems are strictly related to the achievable resolution [2]. Moreover, it can be also viewed as the number of channels while communicating by waves [3].
In this contribution we study the information content of the radiated field for the configuration described below. In particular, we analyze the singular value behavior of the radiation operator and its dependence on the geometrical parameters of the configuration.

2. SVD analysis

We consider the two-dimensional scalar geometry depicted in Fig. 1. The field radiated by an electric current J supported over $S = [-X_s, X_s]$ is observed over two bounded domains. We refer to E_1 as the field observed over $O_1 = [-X_1, X_1]$ and E_2 over the domain $O_2 = [-X_2, X_2]$.

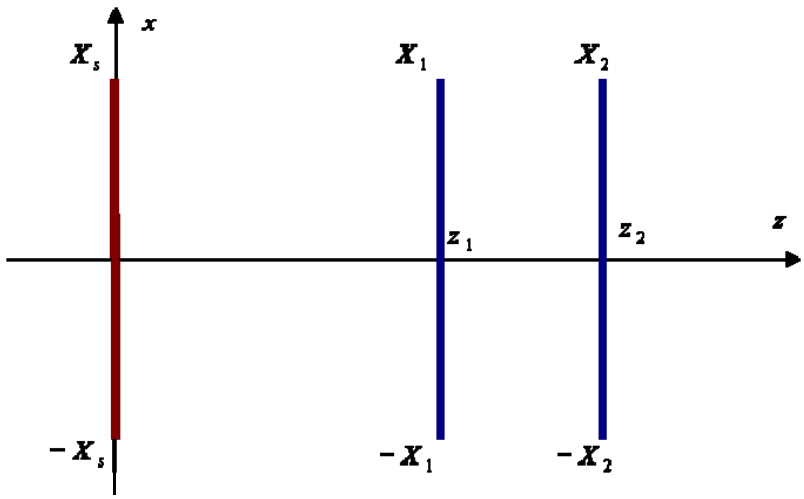

Figure 1 Geometry of the problem

In Fresnel zone, the fields over the domains (normalized to the scalar factor $-\omega\mu/4\sqrt{(2/\beta\pi)}\ \exp(j\pi/4)$) are described as

$$E_1(x_1) = \frac{1}{\sqrt{z_1}} \int_{-X_s}^{X_s} e^{-j\frac{\beta}{2z_1}(x_1-x)^2} J(x)dx \qquad E_2(x_2) = \frac{1}{\sqrt{z_2}} \int_{-X_s}^{X_s} e^{-j\frac{\beta}{2z_2}(x_2-x)^2} J(x)dx \tag{1}$$

If we assume that the source J and the observed fields E_1 and E_2 are complex valued square integrable functions, eq. (1) defines the following operator

$$\mathcal{A}: J \in L^2(S) \rightarrow E = (E_1, E_2) \in L^2(O_1 \times O_2) \tag{2}$$

As stated above, in order to determine the NDF of the radiated field we study the singular value decomposition of the operator $\mathcal{A}$. Accordingly, the NDF will be identified as the number of singular values above a noise dependent threshold. Indeed the singular values represent the *strength* of the connection of a given communication channel between the source and the receiving domain: if this strength is under noise level, the information transmitted using this channel is unreliable [3].
The singular values of the operator may be determined by considering the following eigenvalue problem

$$\gamma_n u_n = \mathcal{A}^+ \mathcal{A}\, u_n \tag{3}$$

where $\mathcal{A}^+$ is the adjoint of the operator $\mathcal{A}$ and γ_ns are the squares of the singular values of the operator $\mathcal{A}$. After simple mathematical handling eq. (3) can be explicitly written as

$$\gamma_n u_n(x) = \lambda \int_{-X_s}^{X_s} \frac{\sin\left[\frac{\beta X_1}{z_1}(x-x')\right]}{\pi(x-x')} e^{j\frac{\beta}{2z_1}(x^2+x'^2)} u(x')dx' + \lambda \int_{-X_s}^{X_s} \frac{\sin\left[\frac{\beta X_2}{z_2}(x-x')\right]}{\pi(x-x')} e^{j\frac{\beta}{2z_2}(x^2+x'^2)} u(x')dx' \tag{4}$$

The eigensystem of operator in eq. (4) is not known in closed form. However, we can simplify the problem by means of an approximation that permits to achieve some analytic results (see [4]).
In the Fresnel zone, $1/z$ is a slowly varying function, so $1/z_1 \approx 1/z_2$. Hence we can assume that the exponentials have almost the same argument and, by posing $\exp(j\,\beta/(2z_1)\ x^2)\,u(x) = \mathfrak{u}(x)$ we obtain the following approximate eigenvalue problem

$$\gamma_n \mathfrak{u}_n(x) = \lambda \int_{-X_s}^{X_s} \frac{\sin\left[\frac{\beta X_1}{z_1}(x-x')\right]}{\pi(x-x')} \mathfrak{u}(x')dx' + \lambda \int_{-X_s}^{X_s} \frac{\sin\left[\frac{\beta X_2}{z_2}(x-x')\right]}{\pi(x-x')} \mathfrak{u}(x')dx' \tag{5}$$

The problem still has no closed form solution, but now the lhs consists of the sum of two integral operators of the type characterized by Slepian and Pollak. Without loss of generality, we assume $\beta X_1/z_1 \geq \beta X_2/z_2$, and define the "spatial" frequency domains: $\Omega_1 = [-\beta X_2/z_2, \beta X_2/z_2]$, $\Omega_2 = [\beta X_2/z_2, \beta X_1/z_1]$, $\Omega_3 = [-\beta X_1/z_1, -\beta X_2/z_2]$. Accordingly, eq. (5) can be rewritten as

$$\gamma_n \mathfrak{u}_n = 2\lambda\, P_S B_{\Omega_1} P_S\ \mathfrak{u}_n + \lambda\, P_S B_{\Omega_2} P_S\ \mathfrak{u}_n + \lambda\, P_S B_{\Omega_3} P_S\ \mathfrak{u}_n \tag{6}$$

where P_S is the spatial limiting projector over S and B_{Ω_1}, B_{Ω_2}, B_{Ω_3} are the band limiting projectors over Ω_1, Ω_2, Ω_3, respectively. The second member consists in the sum of three "Slepian operators" whose eigensystems are well known. In particular we have

$$\lambda\, P_S B_{\Omega_k} P_S\ \mathfrak{u}_n^k = \lambda\, \gamma_n^k\, \mathfrak{u}_n^k \tag{7}$$

where $k = 1,2,3$, $\mathfrak{u}_n^k$ and γ_n^k are the *n*-th eigenfunction and eigenvalue for the *k*-th Slepian operator. Hence $\mathfrak{u}_n^k$s are the *Prolate Spheroidal Wave Functions* (PSWF) of bandwidth Ω_k and γ_n^k's exhibit a single step behavior, i.e.

N^k eigenvalues are close to λ and the remaining ones close to 0 [5]. The position of the step, when the eigenvalues are arranged in non increasing order, is given by the index

$$N^k = \left[\frac{2c_k}{\pi}\right] \tag{8}$$

where $c_k = m\{S\}m\{\Omega_k\}/4$ is the so called *space-bandwidth* product, and $[\cdot]$ denotes the integer part. Furthermore, as shown in [6] we can state that

$$\lambda P_S B_{\Omega_k} P_S \; \mathfrak{U}_n^h \approx 0 \qquad h = 1,2,3 \quad h \neq k \tag{9}$$

Of course, equality in eq. (9) never holds because Slepian operators are positive definite.

From this arises that a good approximation of the eigensystem of problem described in eq. (6) is the mathematical union of the eigensystem of the three Slepian operators (i.e. eigenfunctions set is the union of the three eigenfunctions sets, eigenvalues set is the union of the three eigenvalues set). Hence, we expect that eigenvalues exibit a two step behavior. More in detail we expect that

- $[2c_1/\pi]$ eigenvalues close to 2λ, due to $2\lambda\, P_S B_{\Omega_1} P_s$ operator, that determine a step at the index $N_1 = [2c_1/\pi]$
- $[2c_2/\pi] + [2c_3/\pi]$ eigenvalues close to λ, due to $\lambda\, P_S B_{\Omega_2} P_s$ and $\lambda\, P_S B_{\Omega_3} P_s$ operators, that determine a step at the index

$$N_2 = N_1 + [2c\_2/\pi] + [2c_3/\pi] \tag{10}$$

- beyond N_2 eigenvalues close to 0 due to the three operators.

In particular, the above theory allows to forecast a single step behavior when the integer parts of the last two addends in the expression of N_2 are zeros.

Two examples are reported to verify how well the eigenvalue behavior of the real problem in eq. (4) is approximated by the one of eq. (6).

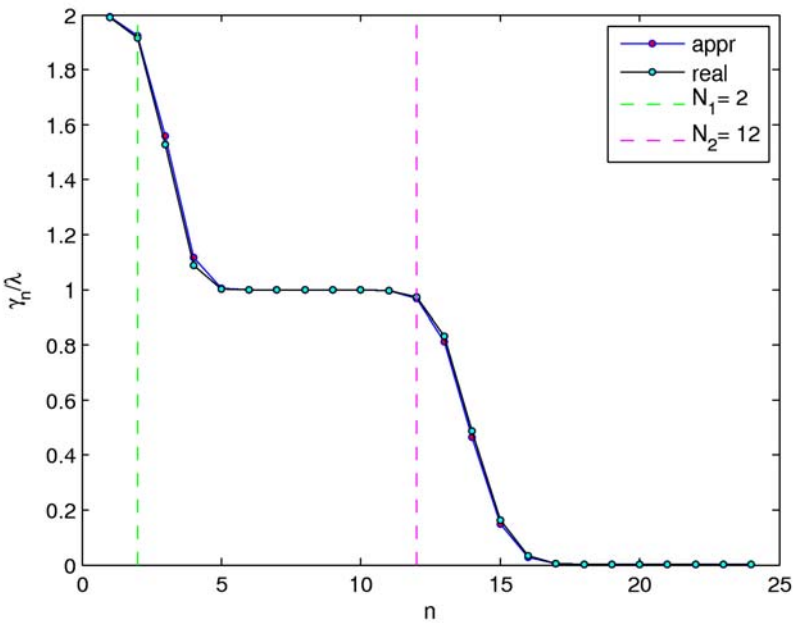

Figure 2 Eigenvalue behavior. Geometry: $X_1 = 20\lambda, X_2 = 20\lambda, X_s = 20\lambda, z_1 = 120\lambda, z_2 = 600\lambda$

In Fig. 1 we see that the two step behavior of the problem is correctly predicted by the approximate model, and the numerical value of the eigenvalues exactly predicted. A good estimate of the eigenvalues is obtained also in the case illustrated in Fig. 2, where a single step is envisaged.

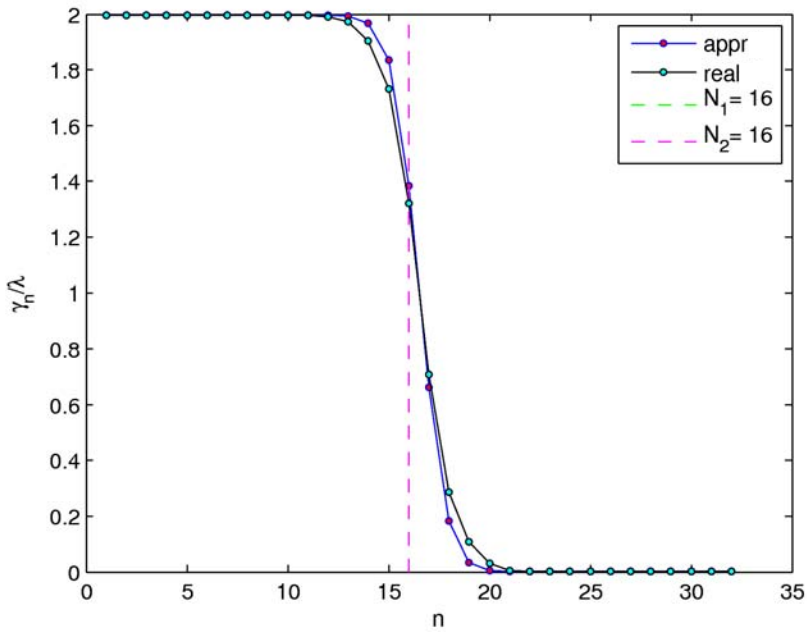

Figure 3 Eigenvalue behavior. Geometry: $X_1 = 20\lambda, X_2 = 20\lambda, X_s = 25\lambda, z_1 = 100\lambda, z_2 = 125\lambda$

5. Conclusion

The problem of determining the NDF of the radiated field observed over two bounded domains has been addressed for a canonical two-dimensional scalar geometry. The study has been achieved in terms of the SVD of the relevant radiation operator. By means of analytical arguments the singular value behaviour and hence the NDF have been predicted in terms of the parameters of the configurations.
The approach followed in this work may be easily generalized to the case of more than two observation domains and to the three-dimensional case of domains that have an extension also along the *z* axis.

7. References

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