A New Value Picking Regularization Method applied to the Electromagnetic Inverse Scattering Problem

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Abstract

This paper presents a new regularization strategy, denoted as Value Picking or VP regularization, which is designed for optimization problems with a large number of optimization variables for which the solution consists of a limited number of different values. When applied to the electromagnetic inverse scattering problem, this regularization favors complex permittivity profiles comprising only a limited number of unknown permittivity values, without a priori knowledge on their spatial distribution. The proposed VP regularizing function only uses an upper estimate on the number of such values. A key ingredient to the VP regularizing function is the choice function, which in general is a real valued function of P variables, but is introduced in this paper only in two dimensions. In this paper, the VP regularizing function is added to the least squares data fit and a Gauss-Newton optimization strategy with line search is applied to minimize the resulting cost function. A 3D reconstruction from simulated noisy data illustrates the effectiveness of the VP regularization scheme.

1. Introduction

It is well known that the exact inverse electromagnetic scattering problem – i.e. the quantitative reconstruction of the complex permittivity of a scatterer from measurements of the scattered field for a number of known incident fields – is ill-posed. The main problem in this respect is instability: in the presence of noise on the data, the reconstructions tend to be very different from the actual permittivity profile. The non-linear inverse scattering problem is usually solved by iteratively minimizing a cost function. Most popular is the least squares data fit cost function

$$F^{LS}(\boldsymbol{\varepsilon}) = \frac{\|\boldsymbol{e}^{\mathrm{scat}}(\boldsymbol{\varepsilon}) - \boldsymbol{e}^{\mathrm{meas}}\|^2}{\|\boldsymbol{e}^{\mathrm{meas}}\|^2} \tag{1}$$

where $\boldsymbol{e}^{\mathrm{meas}}$ and $\boldsymbol{e}^{\mathrm{scat}}$ are vectors that contain respectively the scattered field measurements and the simulated scattered field from a discretized permittivity profile, which is represented by the N-dimensional vector $\boldsymbol{\varepsilon}$. Even when this cost function has a unique global minimum under ideal circumstances, noise on the data introduces ambiguity, i.e. more than one permittivity vector $\boldsymbol{\varepsilon}$ corresponds to a data fit on the noiselevel. To select one acceptable permittivity profile from all these candidates, a regularization term is added to the least squares cost function, which tries to enforce some constraints on the reconstructed profile that are based on a priori knowledge concerning the actual permittivity profile. This approach seems to be more justified than applying regularization only to some linear subproblems in the minimization of F^{LS} as is often seen in literature. Several authors have used a smoothing constraint [1,2] to eliminate the large voxel-to-voxel fluctuations that characterize most of the unwanted permittivity profiles on or beneath the noise level, but when reconstructing piecewise constant permittivity profiles, this kind of regularization is not optimal. Edge preserving regularization strategies have been developed [3], but in this paper we propose a new strategy, the Value Picking or VP regularization, that can be used when the permittivity profile consists of a limited number of different permittivities, or is close to such a profile. It does not use knowledge of where these permittivities are located (or whether they are clustered in piecewise constant areas, as is assumed in edge preserving regularization), but only requires an upper estimate of the number of different values. Moreover, lower and upper bounds on their real and imaginary parts can be easily imposed. The method can be incorporated in voxel-based inversion strategies as a common additive regularization term. In the rest of this paper, the choice function will be introduced in section 2, after which the VP regularizing function is constructed in section 3. The optimization strategy is outlined in section 4 and in section 5 a 3D reconstruction from simulated noisy data is presented to demonstrate the effectiveness of the proposed regularization.

2. The Choice Function

The P-dimensional choice function $f_P(u_1,\ldots,u_P)$ is a fully symmetric positive function of P non-negative variables $(u_1,\ldots,u_P)$. Its most important properties are that $f_P=0$ if and only if at least one of its arguments is zero and that if k of its arguments are much larger than the remaining $P-k$ arguments, the P-dimensional choice function reduces to the $(P-k)$-dimensional choice function evaluated in the smaller arguments. Although it is possible to construct a choice function in any dimension, in this paper we only consider the choice functions in one and two dimensions

$$f_1(u_1)=u_1,\quad f_2(u_1,u_2)=\frac{u_1u_2}{u_1+u_2}, \tag{2}$$

for which the above mentioned properties can be directly verified. Moreover, introducing the functions $B_{21}(u_1,u_2)$ and $B_{22}(u_1,u_2)$, defined by

$$B_{21}(u_1,u_2)=\frac{u_2^2}{(u_1+u_2)^2},\quad B_{22}(u_1,u_2)=\frac{u_1^2}{(u_1+u_2)^2}, \tag{3}$$

it can be proven that

$$\frac{\partial f_2}{\partial u_1}(u_1,u_2)=B_{21}(u_1,u_2),\quad \frac{\partial f_2}{\partial u_2}(u_1,u_2)=B_{22}(u_1,u_2), \tag{4}$$

$$f_2(u_1,u_2)=B_{21}(u_1,u_2)u_1+B_{22}(u_1,u_2)u_2, \tag{5}$$

$$f_2(u_1,u_2)\le B_{21}(u_{1,0},u_{2,0})u_1+B_{22}(u_{1,0},u_{2,0})u_2,\quad \forall(u_1,u_2)\in\mathbf{R}_+^2, \tag{6}$$

where $u_{1,0}$ and $u_{2,0}$ are two arbitrary non-negative numbers.

3. The Value Picking Regularizing Function

To reconstruct a permittivity profile that consists of at the most P different complex permittivity values, we propose to minimize the following cost function:

$$F(\boldsymbol{\varepsilon},\boldsymbol{c})=F^{LS}(\boldsymbol{\varepsilon})+\gamma F^{VP}(\boldsymbol{\varepsilon},\boldsymbol{c}), \tag{7}$$

where γ is a positive regularization parameter and the VP function F^{VP} is given by

$$F^{VP}(\boldsymbol{\varepsilon},\boldsymbol{c})=\frac{1}{N}\sum_{n=1}^{N}f_P(|\epsilon_n-c_1|^2,\ldots,|\epsilon_n-c_P|^2), \tag{8}$$

where the complex variables c_p are called the VP values. Furthermore $c_P=\epsilon_b$ (ϵ_b is the background permittivity) and is collected together with the other $P-1$ unknown VP values in the vector $\boldsymbol{c}$. If we restrict ourselves again to $P=2$, (8) can be rewritten using (5) as

$$F^{VP}(\boldsymbol{\varepsilon},\boldsymbol{c})=\frac{1}{N}\sum_{n=1}^{N}\left[b_{1,n}(\boldsymbol{\varepsilon},\boldsymbol{c})|\epsilon_n-c_1|^2+b_{2,n}(\boldsymbol{\varepsilon},\boldsymbol{c})|\epsilon_n-c_2|^2\right], \tag{9}$$

which can be seen as a weighted sum of the penalty functions $|\epsilon_n-c_p|^2$ where the weights are calculated with the weight functions defined in (3):

$$b_{p,n}(\boldsymbol{\varepsilon},\boldsymbol{c})=B_{2p}(|\epsilon_n-c_1|^2,|\epsilon_n-c_2|^2),\quad \text{for } p=1 \text{ or } p=2. \tag{10}$$

This regularization function, when minimized, tries to enforce equality of every optimization variable ϵ_n with a VP value close to ϵ_n, and disregards the VP values that are clearly farther away from ϵ_n. Indeed, from (10) and (3) it can be seen that if in the course of the minimization $|\epsilon_n-c_1|^2\gg|\epsilon_n-c_2|^2$, for a certain index n, then

$$\begin{aligned}&b_{1,n}(\boldsymbol{\varepsilon},\boldsymbol{c})|\epsilon_n-c_1|^2\to 0\quad\text{and}\\&b_{2,n}(\boldsymbol{\varepsilon},\boldsymbol{c})|\epsilon_n-c_2|^2\to|\epsilon_n-c_2|^2,\end{aligned} \tag{11}$$

such that only the difference between ϵ_n and c_2 is penalized. On the other hand, as long as $|\epsilon_n - c_1|^2 \approx |\epsilon_n - c_2|^2$, we have

$$b_{1,n}(\boldsymbol{\varepsilon}, \boldsymbol{c})|\epsilon_n - c_1|^2 + b_{2,n}(\boldsymbol{\varepsilon}, \boldsymbol{c})|\epsilon_n - c_2|^2 \approx \frac{1}{4}|\epsilon_n - c_1|^2 + \frac{1}{4}|\epsilon_n - c_2|^2, \tag{12}$$

which means that no choice between c_1 and c_2 is made. Furthermore, both terms in (12) have smaller weight than in the case where a choice for c_1 or c_2 *is* made. As a result, the VP regularization does not force a decision too soon if the data fit does not provide enough driving force for it.

4. Minimization of the Cost Function

The cost function (7) is minimized by alternately updating the permittivity profile and the VP values. To update the permittivity vector in iteration k, starting from $(\boldsymbol{\varepsilon}_k, \boldsymbol{c}_k)$, the modified cost function

$$F^Q(\boldsymbol{\varepsilon}, \boldsymbol{c}) = F^{LS}(\boldsymbol{\varepsilon}) + \gamma Q^{VP}(\boldsymbol{\varepsilon}, \boldsymbol{c}), \tag{13}$$

is considered, with

$$Q^{VP}(\boldsymbol{\varepsilon}, \boldsymbol{c}) = \frac{1}{N}\sum_{n=1}^{N}\left[w_{1,n}|\epsilon_n - c_1|^2 + w_{2,n}|\epsilon_n - c_2|^2\right], \tag{14}$$

where the weights $w_{1,n} = b_{1,n}(\boldsymbol{\varepsilon}_k, \boldsymbol{c}_k)$ and $w_{2,n} = b_{2,n}(\boldsymbol{\varepsilon}_k, \boldsymbol{c}_k)$ are calculated in $(\boldsymbol{\varepsilon}_k, \boldsymbol{c}_k)$ and then kept fixed. Based on this modified cost function, which touches with the actual cost function F in $(\boldsymbol{\varepsilon}_k, \boldsymbol{c}_k)$ (because of (4) and (5)) and which lies above F in any other point (because of (6)), the following Gauss-Newton update direction [1] for the permittivity vector $\boldsymbol{\varepsilon}$ is calculated:

$$\boldsymbol{s}_k = \left(\boldsymbol{J}_k^H \boldsymbol{J}_k + \gamma\|\boldsymbol{e}^{\text{meas}}\|^2 \boldsymbol{\Sigma}_k\right)^{-1}\left[\boldsymbol{J}_k^H\left(\boldsymbol{e}^{\text{scat}}(\boldsymbol{\varepsilon}_k) - \boldsymbol{e}^{\text{meas}}\right) + \gamma\|\boldsymbol{e}^{\text{meas}}\|^2 \boldsymbol{\Omega}_k^*\right]. \tag{15}$$

In (15), $\boldsymbol{J}_k$ is the jacobian matrix which contains the derivatives of the scattered field components with respect to the optimization variables, calculated for $\boldsymbol{\varepsilon}_k$, and the elements of the matrix $\boldsymbol{\Sigma}_k$ and the vector $\boldsymbol{\Omega}_k$ respectively are given by

$$[\boldsymbol{\Omega}_k]_n = \frac{1}{N}\left[w_{1,n}(\epsilon_{k,n} - c_{k,1})^* + w_{2,n}(\epsilon_{k,n} - c_{k,2})^*\right], \tag{16}$$

$$[\boldsymbol{\Sigma}_k]_{nm} = \delta_{nm}\frac{1}{N}\left[w_{1,n} + w_{2,n}\right]. \tag{17}$$

The update direction (15) is used as a search direction along which a line search algorithm locates the next permittivity vector $\boldsymbol{\varepsilon}_{k+1}$ for fixed c_k. Note that (15) always is a descent direction with respect to the actual cost function F and that the line search is performed on F and not on F^Q.

To update the VP value c_1 (c_2 is kept fixed and equal to ϵ_b) starting from the current iterate $(\boldsymbol{\varepsilon}_{k+1}, \boldsymbol{c}_k)$, the function (14) with weights $w_{1,n} = b_{1,n}(\boldsymbol{\varepsilon}_{k+1}, \boldsymbol{c}_k)$ and $w_{2,n} = b_{2,n}(\boldsymbol{\varepsilon}_{k+1}, \boldsymbol{c}_k)$ is minimized for c_1, keeping $w_{1,n}$, $w_{2,n}$ and $\boldsymbol{\varepsilon}_{k+1}$ fixed. The weights $w_{1,n}$ and $w_{2,n}$ are updated next and (14) is minimized again. This cycle is repeated until c_1 remains stable. Since (14) is a quadratic function in c_1, its minimization is very simple and upper and lower bounds on the real and imaginary part of c_1 can easily be incorporated in this quadratic optimization problem.

5. Numerical Example

To demonstrate the effect of the VP regularization, we consider 2 homogeneous cubes, one with side λ_b (= the background wavelength) and one with side $0.3\lambda_b$. Both cubes have relative permittivity 2 and the background medium is free space ($\epsilon_b = \epsilon_0$, the permittivity of vacuum). The center of the large cube is $(0.3\lambda_b, 0.3\lambda_b, 0.3\lambda_b)$ and the small cube is centered on $(-0.45\lambda_b, -0.45\lambda_b, -0.45\lambda_b)$. These scatterers are illuminated subsequently from 72 positions on a sphere with radius $2\lambda_b$ by elementary dipole sources with θ- and ϕ- polarizations. For every such illumination, the scattered field is collected in every dipole position along the θ- and ϕ- directions. The data are simulated numerically and 30 dB gaussian noise is added afterwards. The cell size of the permittivity grid is $0.1\lambda_b$ and the investigation domain is a cube with side $2\lambda_b$. Figure 1 shows the real part of the permittivity in two horizontal slices through the investigation domain for the exact profile, for the reconstruction with VP regularization and for the reconstruction

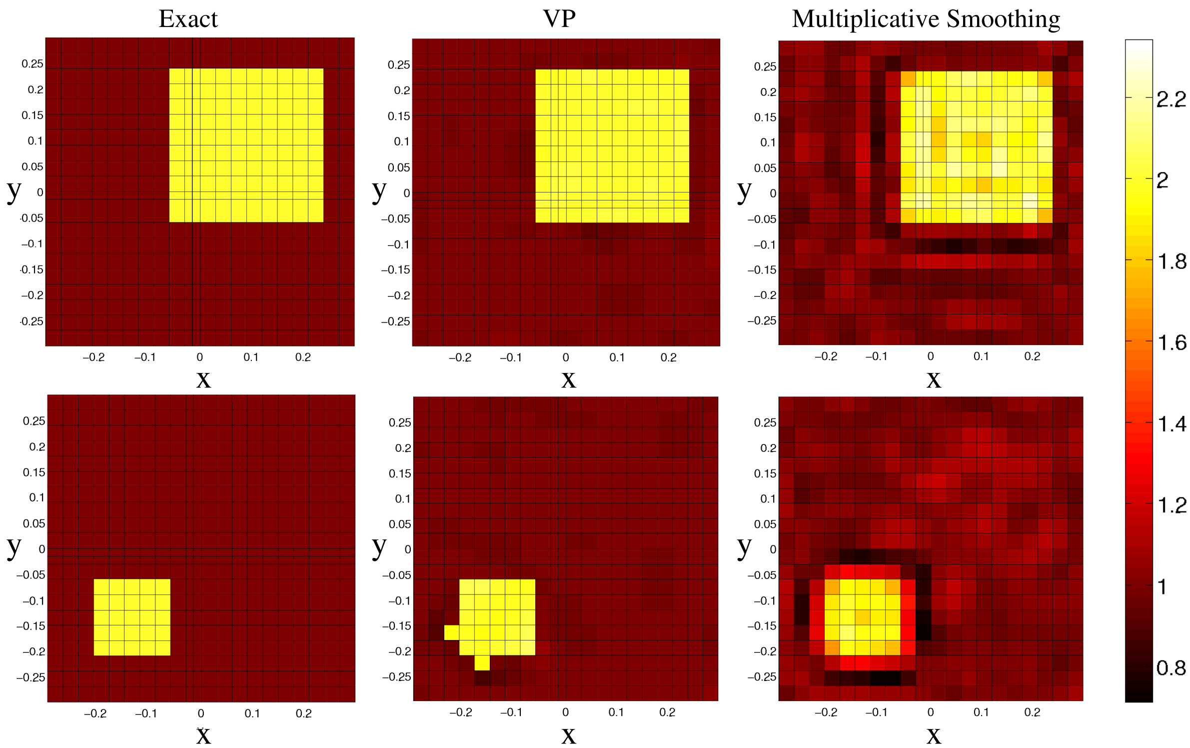

Figure 1: Plots of the real part of the permittivity profile in two horizontal slices ($z = 0.25\lambda_b$ (top row) and $z = -0.45\lambda_b$ (bottom row)) through the investigation domain. The columns, from left to right, correspond to the exact permittivity profile, the reconstruction with VP regularization and the reconstruction with a multiplicative smoothing regularization.

with a multiplicative smoothing regularization as described in [1]. Both reconstructions correspond to a data fit on the noise level but clearly differ in reconstruction quality. The result of the VP regularization is very close to a piecewise homogeneous profile with only 2 different permittivities and compared to the exact profile only shows some small artifacts with the dimensions of one voxel. The optimization ended with $c_1 = 2.0054 - 0.0003j$. The result of the smoothing regularization is more heterogeneous, although it provides a good reconstruction of the object shape.

6. Conclusion

For permittivity profiles that consist of a limited number of different permittivity values, the newly proposed VP regularization strategy is able to provide better reconstructions than a smoothing regularization in case of noisy data, because it incorporates more correct a priori knowledge on the scatterers.

7. References

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