Reduced Rayleigh equations for the scattering of s-polarized light from and its transmission through a film with two one-dimensional rough surfaces

T. A. Leskova[1] and A. A. Maradudin[2]
[1] *Department of Physics and Astronomy, University of California, Irvine, CA 92697, USA (tleskova@uci.edu)*
[2] *Department of Physics and Astronomy, University of California, Irvine, CA 92697, USA (aamaradu@uci.edu)*

We consider s-polarized light incident on a structure consisting of a medium characterized by a dielectric constant ϵ_1 in the region $x_3 > \zeta_1(x_1)$; a film characterized by ϵ_2 in the region $-H+\zeta_2(x_1) < x_3 < \zeta_1(x_1)$; and a substrate characterized by ϵ_3 in the region $x_3 < -H+\zeta_2(x_1)$. The surface profile functions $\zeta_{1,2}(x_1)$are differentiable single-valued functions of x_1. The light is incident from the region $x_3 > \zeta_1(x_1)$, and the plane of incidence is the x_1x_3 plane. Single integral equations for the scattering amplitude and for the transmission amplitude are derived.

I. INTRODUCTION

The theoretical study of the scattering of light from a one-dimensional rough surface is significantly simplified if the electromagnetic field within the scattering medium can be eliminated from consideration, so that it is only the field in the medium of incidence that has to be calculated. The method of reduced Rayleigh equations [1] effects this elimination, and yields a single integral equation for the scattering amplitude, rather than the pair of coupled integral equations for the scattering amplitudes obtained from a straightforward application of the Rayleigh method to the scattering problem. It is desirable to extend this simplification to the study of the scattering of light from, and its transmission through, a film bounded by two one-dimensional rough surfaces. In Ref. [2] a pair of coupled integral equations for the scattering and transmission amplitudes was derived for such a structure. In this paper we simplify the problem even more by deriving a single integral equation for the scattering amplitude and for the transmission amplitude for this structure. Due to limitations on space we present here only the derivations for incident light of s polarization. The corresponding derivations for p-polarized light will be presented elsewhere.

II. THE PHYSICAL SYSTEM STUDIED

The system we consider in this work consists of a medium characterized by a dielectric constant ϵ_1 in the region $x_3 > \zeta_1(x_1)$; a film characterized by a dielectric constant ϵ_2 in the region $-H+\zeta_2(x_1) < x_3 < \zeta_1(x_1)$; and a substrate characterized by a dielectric constant ϵ_3 in the region $x_3 < -H_2+\zeta_2(x_1)$. Each of the surface profile functions $\zeta_{1,2}(x_1)$ is assumed to be a single-valued function of x_1 that is differentiable. The film is illuminated from the region $x_3 > \zeta_1(x_1)$ by s-polarized light of frequency ω, whose plane of incidence is the x_1x_3 plane.

III. EQUATIONS FOR THE SCATTERING AND TRANSMISSION AMPLITUDES

The single nonzero component of the electric field in this structure is given by

$$E_2(x_1,x_3|\omega) \;=\; \exp[ikx_1 - i\alpha_1(k)x_3] + \int_{-\infty}^{\infty} \frac{dq}{2\pi} R(q|k)\exp[iqx_1 + ia_1(q)x_3] \tag{1}$$

in the region $x_3 > \zeta(x_1)_{max}$;

$$E_2(x_1,x_3|\omega) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\exp(iqx_1)\{A_1(q|k)\exp[i\alpha_2(q)x_3] + A_2(q|k)\exp[-i\alpha_2(q)x_3]\} \tag{2}$$

in the region $-H+\zeta_2(x_1)<x_3<\zeta(x_1)$;

$$E_2(x_1,x_3|\omega)=\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}T(q|k)\exp[iqx_1-i\alpha_3(q)x_3] \tag{3}$$

in the region $x_3<-H+\zeta_2(x_1)_{min}$, where $\alpha_i(q)=[\epsilon_i(\omega/c)^2\ -q^2]^{\frac{1}{2}}$, with $Re\alpha_i(q)>0$, $Im\alpha_i(q)>0$.

The boundary conditions at the interface $x_3=\zeta_1(x_1)$ are

$$\begin{aligned}&\exp[ikx_1-i\alpha_1(k)\zeta_1(x_1)]+\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}R(q|k)\exp[iqx_1+i\alpha_1(q)\zeta_1(x_1)]\\&=\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}\{A_1(q|k)\exp[iqx_1+i\alpha_2(q)\zeta_1(x_1)]+A_2(q|k)\exp[iqx_1-i\alpha_2(q)\zeta_1(x_1)]\}\end{aligned} \tag{4}$$

$$\begin{aligned}&[-\zeta_1'(x_1)k-\alpha_1(k)]\exp[ikx_1-i\alpha_1(k)\zeta_1(x_1)]\\&\quad+\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}[-\zeta_1'(x_1)q+\alpha_1(q)]R(q|k)\exp[iqx_1+i\alpha_1(q)\zeta_1(x_1)]\\&\quad=\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}\{A_1(q|k)[-\zeta_1'(x_1)q+i\alpha_2(q)]\exp[iqx_1+i\alpha_2(q)\zeta_1(x_1)]\\&\qquad+A_2(q|k)[-\zeta_1'(x_1)q-\alpha_2(q)]\exp[iqx_1-i\alpha_2(q)\zeta_1(x_1)]\}.\end{aligned} \tag{5}$$

The boundary conditions at the interface $x_3=-H+\zeta_2(x_1)$ are

$$\begin{aligned}&\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}\{A_1(q|k)\exp[iqx_1+i\alpha_2(q)(-H+\zeta_2(x_1))]+A_2(q|k)\exp[iqx_1-i\alpha_2(q)(-H+\zeta_2(x_1))]\}\\&=\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}T(q|k)\exp[iqx_1-i\alpha_3(q)(-H+\zeta_2(x_1))]\end{aligned} \tag{6}$$

$$\begin{aligned}&\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}\{A_1(q|k)[-\zeta_2'(x_1)q\alpha_2(q)]\exp[iqx_1+i\alpha_2(q)(-H+\zeta_2(x_1))]\\&\quad+A_2(q|k)[-\zeta_2'(x_1)q-\alpha_2(q)]\exp[iqx_1-i\alpha_2(q)(-H+\zeta_2(x_1))]\}\\&\qquad=\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}T(q|k)[-\zeta_2'(x_1)q-\alpha_3(q)]\exp[iqx_1-i\alpha_3(q)(-H+\zeta_2(x_1))].\end{aligned} \tag{7}$$

To obtain a single integral equation satisfied by the scattering amplitude $R(q|k)$ we first multiply Eq. (4) by $[-\zeta_1'(x_1)p+\alpha_2(p)]\ \exp[-ipx_1-i\alpha_2(p)\zeta_1(x_1)]$, multiply Eq. (5) by $\exp[-ipx_1-i\alpha_2(p)\zeta_1(x_1)]$, integrate the resulting equations over x_1, and add them. The result is

$$\begin{aligned}&(\epsilon_2-\epsilon_1)\frac{\omega^2}{c^2}\frac{I_1(\alpha_2(p)+\alpha_1(k)|p-k)}{\alpha_2(p)+\alpha_1(k)}\\&\quad+(\epsilon_2-\epsilon_1)\frac{\omega^2}{c^2}\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{I_1(\alpha_2(p)-\alpha_1(q)|p-q)}{\alpha_2(p)-\alpha_1(q)}R(q|k)=2\alpha_2(p)A_1(p|k).\end{aligned} \tag{8}$$

We next multiply Eq. (4) by $[\zeta_1'(x_1)p+\alpha_2(p)]\exp[-ipx_1+i\alpha_2(p)\zeta_1(x_1)]$, multiply Eq. (5) by $\exp[-ipx_1+i\alpha_2(p)\zeta_1(x_1)]$, integrate the resulting equations over x_1, and add them. The result is

$$(\epsilon_2-\epsilon_1)\frac{\omega^2}{c^2}\frac{I_1(-\alpha_2(p)+\alpha_1(k)|p-k)}{\alpha_2(p)-\alpha_1(k)}$$
$$+(\epsilon_2-\epsilon_1)\frac{\omega^2}{c^2}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{I_1(-\alpha_2(p)-\alpha_1(q)|p-q)}{\alpha_2(p)+\alpha_1(q)}R(q|k)=2\alpha_2(p)A_2(p|k). \tag{9}$$

In Eqs. (8) and (9) we have introduced the function

$$I_1(\gamma|Q)=\int_{-\infty}^{\infty}dx_1\exp[-iQx_1-i\gamma\zeta_1(x_1)]. \tag{10}$$

We continue by multiplying Eq. (6) by $[-\zeta_2'(x_1)p+\alpha_3(p)]\exp[-ipx_1-i\alpha_3(p)\zeta_2(x_1)]$, multiplying Eq. (7) by $\exp[-ipx_1-i\alpha_3(p)\zeta_2(x_1)]$, integrating the resulting equations over x_1, and adding them. The result is

$$(\epsilon_3-\epsilon_2)\frac{\omega^2}{c^2}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\left\{\frac{I_2(\alpha_3(p)-\alpha_2(q)|p-q)}{\alpha_3(p)-\alpha_2(q)}\exp[-i\alpha_2(q)H]A_1(q|k)\right.$$
$$\left.+\frac{I_2(\alpha_3(p)+\alpha_2(q)|p-q)}{\alpha_3(p)+\alpha_2(q)}\exp[i\alpha_2(q)H]A_2(q|k)\right\}=0, \tag{11}$$

where

$$I_2(\gamma|Q)=\int_{-\infty}^{\infty}dx_1\exp[-iQx_1-i\gamma\zeta_2(x_1)]. \tag{12}$$

We substitute Eqs. (8) and (9) into Eq. (11), and obtain

$$\int_{-\infty}^{\infty}\frac{dq}{2\pi}\left\{\frac{I_2(\alpha_3(p)-\alpha_2(q)|p-q)}{\alpha_3(p)-\alpha_2(q)}\frac{\exp[-i\alpha_2(q)H]}{\alpha_2(q)}\frac{I_1(\alpha_2(q)+\alpha_1(k)|q-k)}{\alpha_2(q)+\alpha_1(k)}\right.$$
$$\left.+\frac{I_2(\alpha_3(p)+\alpha_2(q)|p-q)}{\alpha_3(p)+\alpha_2(q)}\frac{\exp[i\alpha_2(q)H]}{\alpha_2(q)}\frac{I_1(-\alpha_2(q)+\alpha_1(k)|q-k)}{\alpha_2(q)-\alpha_1(k)}\right\}$$
$$+\int_{-\infty}^{\infty}\frac{dq}{2\pi}\int_{-\infty}^{\infty}\frac{dr}{2\pi}\left\{\frac{I_2(\alpha_3(p)-\alpha_2(q)|p-q)}{\alpha_3(p)-\alpha_2(q)}\frac{\exp[-i\alpha_2(q)H]}{\alpha_2(q)}\frac{I_1(\alpha_2(q)-\alpha_1(r)|q-r)}{\alpha_2(q)-\alpha_1(r)}\right.$$
$$\left.+\frac{I_2(\alpha_3(p)+\alpha_2(q)|p-q)}{\alpha_3(p)+\alpha_2(q)}\frac{\exp[i\alpha_2(q)H]}{\alpha_2(q)}\frac{I_1(-\alpha_2(q)-\alpha_1(r)|q-r)}{\alpha_2(q)+\alpha_1(r)}\right\}R(r|k)=0. \tag{13}$$

Equation (13) is the reduced Rayleigh equation for the scattering amplitude $R(q|k)$ in s polarization.

To obtain a single integral equation for the transmission amplitude $T(q|k)$ we first multiply Eq. (6) by $[-\zeta_2'(x_1)p+\alpha_2(p)]\exp[-ipx_1-i\alpha_2(p)\zeta_2(x_1)]$, multiply Eq. (7) by $\exp[-ipx_1-i\alpha_2(p)\zeta_2(x_1)]$, integrate the resulting equations over x_1 and add them. In this way we obtain

$$2\alpha_2(p)A_1(p|k)=(\epsilon_2-\epsilon_3)\frac{\omega^2}{c^2}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{I_2(\alpha_2(p)+\alpha_3(q)|p-q)}{\alpha_2(p)+\alpha_3(q)}$$
$$\times\exp[i(\alpha_2(p)+\alpha_3(q))H]T(q|k). \tag{14}$$

We next multiply Eq. (6) by $[\zeta_2'(x_1)p+\alpha_2(p)[\exp[-ipx_1+i\alpha_2(p)\zeta_2(x_1)$, multiply Eq. (7) by $\exp[-ipx_1+i\alpha_2(p)\zeta_2(x_1)]$, integrate the resulting equations over x_1, and subtract the second from the first. The result is

$$2\alpha_2(p)A_2(p|k) \;=\; (\epsilon_3-\epsilon_2)\frac{\omega^2}{c^2}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{I_2(-\alpha_2(p)+\alpha_3(q)|p-q)}{-\alpha_2(p)+\alpha_3(q)}$$
$$\times\exp[-i(\alpha_2(p)-\alpha_3(q))H]T(q|k). \tag{15}$$

We continue by multiplying Eq. (4) by $[\zeta_1'(x_1)p + \alpha_1(p)]\exp[-ip\zeta(x_1) + i\alpha_1(p)\zeta_1(x_1)]$, multiplying Eq. (5) by $\exp[-ipx_1 + i\alpha_1(p)\zeta_1(x_1)]$, integrating the resulting equations over x_1, and subtracting the second from the first. This yields the result

$$\begin{aligned} 2\alpha_1(k)2\pi\delta(p-k) \;=\; (\epsilon_1-\epsilon_2)\frac{\omega^2}{c^2}\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}\Bigg\{&\frac{I_1(-\alpha_1(p)-\alpha_2(q)|p-q)}{\alpha_1(p)+\alpha_2(q)}A_1(q|k) \\ &+\frac{I_1(-\alpha_1(p)+\alpha_2(q)|p-q)}{\alpha_1(p)-\alpha_2(q)}A_2(q|k)\Bigg\}. \end{aligned} \tag{16}$$

Finally, we substitute Eqs. (14) and (15) into Eq. (16), and obtain

$$\begin{aligned} 2\alpha_1(k)2\pi\delta(p-k) = (\epsilon_1-\epsilon_2)(\epsilon_2-\epsilon_3)\left(\frac{\omega}{c}\right)^4\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dr}{2\pi} \\ \times\Bigg\{\frac{I_1(-\alpha_1(p)-\alpha_2(q)|p-q)}{\alpha_1(p)+\alpha_2(q)}\frac{\exp[i(\alpha_2(q)+\alpha_3(r))H]}{2\alpha_2(q)}\frac{I_2(\alpha_2(q)+\alpha_3(r)|q-r)}{\alpha_2(q)+\alpha_3(r)} \\ +\frac{I_1(-\alpha_1(p)+\alpha_2(q)|p-q)}{\alpha_1(p)-\alpha_2(q)}\frac{\exp[-i(\alpha_2(q)+\alpha_3(r))H]}{2\alpha_2(q)}\frac{I_2(-\alpha_2(q)+\alpha_3(r)|q-r)}{\alpha_2(q)-\alpha_3(r)}\Bigg\}T(r|k). \end{aligned} \tag{17}$$

Equation (17) is the reduced Rayleigh equation for the transmission amplitude $T(q|k)$ in s polarization.

IV. CONCLUSIONS

We have shown how the system of four coupled integral equations (4)-(7) from which the amplitude for the scattering of light from, and the amplitude for the transmission of light through, a film bounded by a pair of one-dimensional rough surfaces, can be reduced to a single integral equation for each of these amplitudes alone. Because they are two-dimensional integral equations, their purely numerical solution will be more difficult than the numerical solution of the pair of coupled one-dimensional integral equations for the scattering and transmission amplitudes. However, these equations provide a convenient starting point for the calculations of these amplitudes by means of small-amplitude perturbation theory.

Acknowledgement

This research was supported in part by Army Research Office grant W911NF-06-1-0385.

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