An Efficient T-matrix Method for Analyzing the Electromagnetic Scattering of Multiple PEC Objects

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Introduction

The T-matrix method has been of a great interest in solving the electromagnetic scattering problems since it is firstly discussed in [1]. Later, [2] extends the T-matrix method to an arbitray number of scatterers. The total T-matrix is expressed in terms of the individual T-matrices by an iterative procedure. In order to reduce the computation cost of the total T-matrix, a recursive algorithm is proposed in [3]. Since then, the use of this idea for various structures has been demonstrated [4, 5]. In this paper, we propose firstly an efficient way to obtain a small T-matrix for any arbitrary-shaped conducting scatterer. Based on the T-matrix information for each scatterer, an algorithm considering the interactions of multiple scatterers is then proposed. Since the dimension of each single T-matrix is much smaller than the unknowns of MOM method, the computation cost is greatly reduced for multiple-scatterer problems. Moreover, due to the incidence independent property of the T-matrices, they don't need the recalculation for different incident cases. This further reduces the cost.

Formulations

A dyadic Green's function in EM can be written as

$$\overline{\mathbf{G}}(\mathbf{r}_j, \mathbf{r}_i) = \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) g(\mathbf{r}_j, \mathbf{r}_i) \tag{1}$$

It can be expanded by vector wave functions in spherical coordinates [6]

$$\overline{\mathbf{G}}(\mathbf{r}_j, \mathbf{r}_i) = ik \sum_{l=1}^{\infty} \sum_{l=-m}^{m} \frac{1}{l(l+1)} [\mathbf{M}_{lm}(k, \mathbf{r}_{js}) \Re g \hat{\mathbf{M}}_{lm}(k, \mathbf{r}_{is})) + \mathbf{N}_{lm}(k, \mathbf{r}_{js}) \Re g \hat{\mathbf{N}}_{lm}(k, \mathbf{r}_{is}))] \tag{2}$$

where, $\mathbf{r}_j - \mathbf{r}_i = \mathbf{r}_{js} - \mathbf{r}_{is}$, $|\mathbf{r}_{js}| < |\mathbf{r}_{is}|$. $\mathbf{M}_{lm}$ and $\mathbf{N}_{lm}$ are vector wave spherical harmonics expressed in terms of spherical Hankel functions and spherical harmonics. $\Re g$ means taking the regular part of the function where the spherical Hankel function is replaced by a spherical Bessel function. Truncating the summation at $l = l_{max}$, then the number of terms involved in (2) is $P = (l_{max} + 1)^2 - 1$. Therefore, (2) can be rewritten in a more compact form as

$$\overline{\mathbf{G}}(\mathbf{r}_j, \mathbf{r}_i) = \overline{\boldsymbol{\psi}}^t(\mathbf{r}_{js})_{3\times 2P} \cdot \Re g \hat{\overline{\boldsymbol{\psi}}}(\mathbf{r}_{is})_{2P\times 3} \tag{3}$$

Here, $\overline{\boldsymbol{\psi}}^t(\mathbf{r}_{js})$ and $\Re g \hat{\overline{\boldsymbol{\psi}}}(\mathbf{r}_{is})$ are matrices composed of ordered $\mathbf{M}_{lm}$ and $\mathbf{N}_{lm}$. They can be further expanded by the vector addition theorem for $\mathbf{r} = \mathbf{r}' + \mathbf{r}''$ and $|\mathbf{r}'| < |\mathbf{r}''|$ as below

$$\overline{\boldsymbol{\psi}}^t(\mathbf{r})_{3\times 2P} = \Re g \overline{\boldsymbol{\psi}}^t(\mathbf{r}')_{3\times 2P} \cdot \overline{\boldsymbol{\alpha}}(\mathbf{r}'')_{2P\times 2P} \tag{4}$$

or,

$$\overline{\boldsymbol{\psi}}^t(\mathbf{r})_{3\times 2P} = \overline{\boldsymbol{\psi}}^t(\mathbf{r}'')_{3\times 2P} \cdot \overline{\boldsymbol{\beta}}(\mathbf{r}')_{2P\times 2P} \tag{5}$$

where $\overline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\beta}}$ are translation operators defined as in [6]. Hence, the dyadic Green's function is expanded into another form by applying (4) to (3),

$$\overline{\mathbf{G}}(\mathbf{r}_j, \mathbf{r}_i) = \Re g\overline{\boldsymbol{\psi}}^t(\mathbf{r}_{js'})_{3\times 2P} \cdot \overline{\boldsymbol{\alpha}}(\mathbf{r}_{s's})_{2P\times 2P} \cdot \Re g\hat{\overline{\boldsymbol{\psi}}}(\mathbf{r}_{is})_{2P\times 3} \tag{6}$$

The scattering solution of a PEC object is known as

$$\mathbf{E}^s(\mathbf{r}) = i\omega\mu \int \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')d\mathbf{r}' \tag{7}$$

By discretizing the PEC object into M RWG basis and using the expansion of dyadic Green's function in (3), (7) can be expanded into

$$\mathbf{E}^s(\mathbf{r}) = \sum_{i=1}^{M} \overline{\boldsymbol{\psi}}^t(\mathbf{r} - \mathbf{r}_i) \cdot \mathbf{M}_{ii} a_i \tag{8}$$

where $\mathbf{M}_{ii}$ is defined as

$$\mathbf{M}_{ii} = i\omega\mu \int_{S_i} \Re g\hat{\overline{\boldsymbol{\psi}}}(\mathbf{r}'_i - \mathbf{r}_i) \cdot \boldsymbol{\Lambda}_i(\mathbf{r}'_i)d\mathbf{r}'_i \tag{9}$$

In the above, $\mathbf{r}_i$ denotes the center of the i-th RWG, $\mathbf{r}'_i$ are the sampling points on the RWG, a_i is the current coefficient. In this way, each basis on the PEC surface is regarded as a subscatterer and its scattered field is expanded into outgoing wave function form. More compactly, (8) can be rewritten in a matrix form,

$$\mathbf{E}^s(\mathbf{r}) = \overline{\boldsymbol{\Psi}}^t(\mathbf{r}) \cdot \overline{\mathbf{M}}_{(1)} \cdot \mathbf{a}_{(1)} \tag{10}$$

$\overline{\boldsymbol{\Psi}}^t(\mathbf{r})$ and $\overline{\mathbf{M}}_{(1)}$ are larger matrices stacked by $\overline{\boldsymbol{\psi}}^t(\mathbf{r} - \mathbf{r}_i)$ and $\mathbf{M}_{ii}$, $i = 1, 2, ..., M$, respectively as below

$$\overline{\boldsymbol{\Psi}}^t(\mathbf{r}) = [\overline{\boldsymbol{\psi}}^t(\mathbf{r} - \mathbf{r}_1), \overline{\boldsymbol{\psi}}^t(\mathbf{r} - \mathbf{r}_2), ..., \overline{\boldsymbol{\psi}}^t(\mathbf{r} - \mathbf{r}_M)], \tag{11}$$

$$\overline{\mathbf{M}}_{(1)} = [\mathbf{M}_{11}, \mathbf{M}_{22}, ..., \mathbf{M}_{MM}]^t. \tag{12}$$

$\mathbf{a}_{(1)}$ is the current coefficient vector. Subscript (1) indicates for one object.

Likewise, the incident field by any source can be expanded. Suppose $\mathbf{J}(\mathbf{r}_s)$ are any kind of sources located at $\mathbf{r}_s$, then the incident field at $\mathbf{r}'_i$ is

$$\mathbf{E}^i(\mathbf{r}'_i) = i\omega\mu \int \overline{\mathbf{G}}(\mathbf{r}'_i - \mathbf{r}_s) \cdot \mathbf{J}(\mathbf{r}_s)d\mathbf{r}_s \tag{13}$$

Applying (6) to the dyadic Green's function above, one can get the incident field on the i-th RWG as

$$\mathbf{E}^i(\mathbf{r}'_i) = \Re g\overline{\boldsymbol{\psi}}^t(\mathbf{r}'_i - \mathbf{r}_i) \cdot \overline{\boldsymbol{\alpha}}_{is} \cdot \mathbf{a}_s \tag{14}$$

So far both the incident and scattered fields have been expanded into the wave function form. Therefore, a T-matrix can be defined as below to express the linear relationship between them, i.e.

$$\mathbf{E}^s(\mathbf{r}) = \overline{\boldsymbol{\Psi}}^t(\mathbf{r}) \cdot \overline{\mathbf{T}}_{(1)} \cdot \overline{\boldsymbol{\alpha}}_s \cdot \mathbf{a}_s \tag{15}$$

where $\overline{\boldsymbol{\alpha}}_s$ is a larger matrix stacked by $\overline{\boldsymbol{\alpha}}_{is}$.In order to calculate the T-matrix, apply the boundary condition on the PEC surface first. That is,

$$0 = \int_{S_i} \boldsymbol{\Lambda}_i(\mathbf{r}'_i) \cdot \{\mathbf{E}^i(\mathbf{r}'_i) + \mathbf{E}^s(\mathbf{r}'_i)\}d\mathbf{r}'_i \tag{16}$$

$i = 1, 2, ..., M$. Replacing $\mathbf{E}^i$ and $\mathbf{E}^s$ with (14) and (7) respectively, we have

$$-\overline{\mathbf{N}}_{(1)} \cdot \overline{\boldsymbol{\alpha}}_s \cdot \mathbf{a}_s = \overline{\mathbf{S}} \cdot \mathbf{a}_{(1)} \tag{17}$$

where $\overline{\mathbf{S}}$ is the same matrix as that of MOM method. $\overline{\mathbf{N}}_{(1)}$ is a matrix with diagonal subblocks defined as

$$\mathbf{N}_{ii} = \int_{S_i} \mathbf{\Lambda}_i(\mathbf{r}'_i) \cdot \Re g \overline{\boldsymbol{\psi}}^t(\mathbf{r}'_i - \mathbf{r}_i) d\mathbf{r}'_i \tag{18}$$

(17) can be applied to the definition of $\overline{\mathbf{T}}_{(1)}$. Consequently, we get

$$\overline{\mathbf{T}}_{(1)(2PM\times 2PM)} = -\overline{\mathbf{M}}_{(1)(2PM\times M)} \cdot \overline{\mathbf{S}}^{-1}_{(M\times M)} \cdot \overline{\mathbf{N}}_{(1)(M\times 2PM)} \tag{19}$$

The way to derive the T-matrix so far follows the same idea proposed in [5]. Notice that it is derived in terms of each basis on the PEC surface, so $\overline{\mathbf{T}}_{(1)}$ has the dimension at least the same as that of MOM method, which is M^2. Actually, if the number of wave functions used for the field expansion of each basis is $2P$, then the dimension of $\overline{\mathbf{T}}_{(1)}$ is $(2PM)^2$, which is even larger than that of MOM method. In order to reduce the dimension of $\overline{\mathbf{T}}_{(1)}$, we apply the addition theorem (5) to $\overline{\boldsymbol{\psi}}^t(\mathbf{r}-\mathbf{r}_i)$ in $\overline{\mathbf{\Psi}}^t(\mathbf{r})$ above. In this way, the scattering center is pushed from the centers of all the basis to the center of the entire object. Therefore, the scattered field can be recalculated as

$$\begin{aligned}\mathbf{E}^s(\mathbf{r}) &= \overline{\mathbf{\Psi}}^t(\mathbf{r}-\mathbf{r}_0) \cdot \overline{\boldsymbol{\beta}}_0 \cdot \overline{\mathbf{T}}_{(1)} \cdot \overline{\boldsymbol{\alpha}}_s \cdot \mathbf{a}_s \\ &= \overline{\mathbf{\Psi}}^t(\mathbf{r}-\mathbf{r}_0) \cdot \tilde{\overline{\mathbf{T}}}_{(1)} \cdot \mathbf{a}_s\end{aligned} \tag{20}$$

Here, $\mathbf{r}_0$ is the center of the object. $\tilde{\overline{\mathbf{T}}}_{(1)}$ is defined as

$$\tilde{\overline{\mathbf{T}}}_{(1)(2P\times 2P)} = \overline{\boldsymbol{\beta}}_{0(2P\times 2PM)} \cdot \overline{\mathbf{T}}_{(1)(2PM\times 2PM)} \cdot \overline{\boldsymbol{\alpha}}_{s(2PM\times 2P)} \tag{21}$$

Since P can be taken as small numbers at low frequency, the dimension of $\tilde{\overline{\mathbf{T}}}_{(1)}$ is greatly reduced in comparison with $\overline{\mathbf{T}}_{(1)}$. Because $\tilde{\overline{\mathbf{T}}}_{(1)}$ characterizes the scattering properties of the scatterer and depends on the scatterer only, but not on the incident field. This makes it very useful for the calculation of the multiple-scatterer problems when we know the properties of each subscatterer. We will see this in the next part.

Suppose there are N PEC objects, B_m, $m = 1, 2, ..., N$. The boundary of B_m is S_m. When a given wave is incident upon the N objects, the total field can be calculated by

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \mathbf{E}^i(\mathbf{r}) + \mathbf{E}^s(\mathbf{r}) \\ &= \Re g\overline{\mathbf{\Psi}}^t(\mathbf{r}-\mathbf{r}_m) \cdot \overline{\boldsymbol{\alpha}}_{ms} \cdot \mathbf{a}_s + \overline{\mathbf{\Psi}}^t(\mathbf{r}-\mathbf{r}_m) \cdot \mathbf{b}_m + \sum_{n=1,n\neq m}^{N} \Re g\overline{\mathbf{\Psi}}^t(\mathbf{r}-\mathbf{r}_m) \cdot \overline{\boldsymbol{\alpha}}_{mn} \cdot \mathbf{b}_n\end{aligned} \tag{22}$$

where $\mathbf{b}_m$ and $\mathbf{b}_n$ are the wave function coefficients for the m-th and n-th object respectively, $\mathbf{r}_m$ and $\mathbf{r}_n$ are the centers of the m-th and n-th object. Here we expand the incident and scattered fields about the center of the m-th scatterer $\mathbf{r}_m$. Then by the definition of the T matrix for a single object and the boundary condition on each object, one can get the matrix equation for $\mathbf{b}_m$ as

$$\mathbf{b}_m = \tilde{\overline{\mathbf{T}}}_{m(1)} \cdot (\mathbf{a}_s + \sum_{n=1,n\neq m}^{N} \overline{\boldsymbol{\alpha}}^{-1}_{ms} \cdot \overline{\boldsymbol{\alpha}}_{mn} \cdot \mathbf{b}_n) \tag{23}$$

$m = 1, 2, ..., N$. In (22), vector addition theorem below has been applied

$$\overline{\mathbf{\Psi}}^t(\mathbf{r}-\mathbf{r}_n) = \Re g\overline{\mathbf{\Psi}}^t(\mathbf{r}-\mathbf{r}_m) \cdot \overline{\boldsymbol{\alpha}}_{mn} \tag{24}$$

with condition of $|\mathbf{r}-\mathbf{r}_m| < |\mathbf{r}_m - \mathbf{r}_n|$. This enforces the method with the requirement that any two objects can't have intersection with each other. After solving for $\mathbf{b}_m$, the scattering wave out of region $S_1 \cup S_2 \cup ... \cup S_N$ can be calculated as

$$\mathbf{E}^s(\mathbf{r}) = \sum_{m=1}^{N} \overline{\mathbf{\Psi}}(\mathbf{r}-\mathbf{r}_m) \cdot \mathbf{b}_m \tag{25}$$

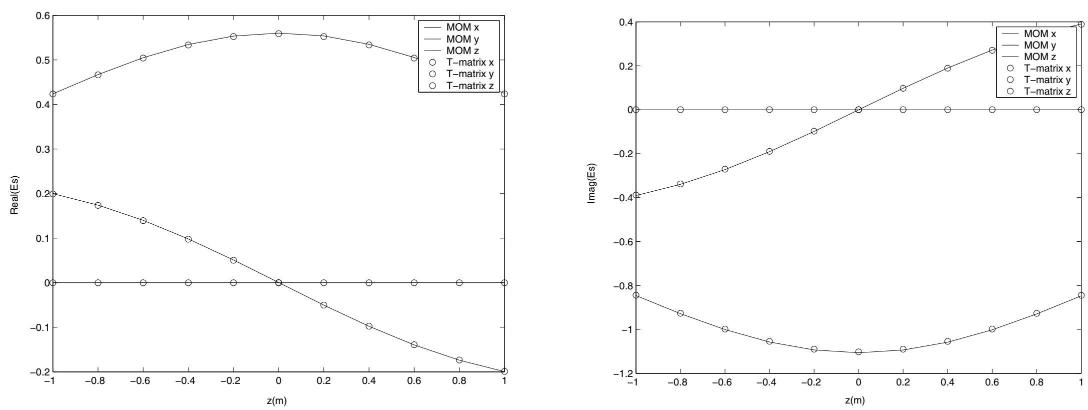
The computation cost for the matrix equation (23) is $(2PN)^3$, which is much smaller than $(MN)^3$ of the MOM method.

Numerical Results

To test the efficiency of the algorithms proposed, the scattering solution by two PEC spheres at the frequency of 0.003 GHz is compared with that of MOM method. Both of the spheres have radius of 1 m and have the centers located at (0, 0, 0) and (-3.0, 0, 0) respectively. Both of them are discretized into 444 patches. A dipole located at x = 20.0 m and radiating in $-\hat{x}$ direction with the magnitude of 100 is used as the excitation. P is taken as 3. Figure 1 shows the electric field outside the two spheres at x = -6.0 m(0.06 λ away from the origin), y = 0 m, z $\in$ [-2.0, 2.0] m. We can see a good agreement between the scattered field of the proposed method and MOM method. As for the number of unknowns, the proposed method has 4P which is 12 and MOM method has 888 for this case, which means that the unknown reduction is 98.6%.

Conclusion

Using the new T-matrix calculation idea for PEC object allows us to enjoy the benefit of cost reduction in solving the multiple-scatterer problems. Besides, compared to conventional MOM method, the T-matrix method can easily handle different incident cases. Applications to more complicated structures will be investigated in the future.



(a) (b)

Figure 1: Scattered field of two PEC spheres.(a):Real part, (b):Imaginary part.

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