A Generalized Scattering Matrix for Arbitrarily Shaped 2D Scatterers

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Abstract

In this paper, by means of a convenient re-interpretation of the Method of Auxiliary Sources, we propose a semi-analytical approach for the evaluation of a *generalized* scattering matrix (G-SM) for arbitrarily shaped 2D objects. A simple procedure based on the Singular Value Decomposition of the relevant integral operators is also outlined to fix the proper dimension of the G-SM. As an example, the scattered field from a non-canonical inhomogeneous dielectric object is computed.

1. Introduction

The scattering matrix is a computationally inexpensive tool for evaluating the field scattered from dielectric and/or metallic objects. As a matter of fact, provided one can represent the incident field in terms of a suitable expansion of coefficients $\boldsymbol{a}$, the scattering matrix $\boldsymbol{S}$ allows to directly calculate the corresponding coefficients of the scattered field via the solution of the linear system, $\boldsymbol{b} = \boldsymbol{S}\,\boldsymbol{a}$. However, while this expression has a general validity, its practical applicability is very limited, as the scattering matrix can be computed in a closed form only when the scatterer has a canonical geometry [1]. As this is not the usual case, alternative (numerical) techniques are thus exploited in practical applications. To overcome this limitation, in this paper we introduce a *generalized* scattering matrix (G-SM) for two-dimensional arbitrarily shaped objects and provide a procedure for its evaluation. In particular, to achieve the "input-output" relationship of a generic object, we take advantage of a suitable re-interpretation of the Method of Auxiliary Sources (MAS) [2]. In MAS, the boundary problem is solved by properly superimposing a set of elementary solutions (the auxiliary sources) located on surfaces conformal to that of the scatterer. As such, MAS is a convenient choice to handle non-canonical shapes within a finite dimensional framework, defined by the number of auxiliary sources. Then, the limited bandwidth properties of the scattered fields are exploited to determine the size of the matrix in terms of the required number of auxiliary sources.

2. The *generalized* scattering matrix (G-SM)

Let us consider the scalar two-dimensional scattering problem sketched in Fig. 1, where a generic scatterer Ω invariant along the z axis is illuminated by a z-polarized electric incident field. For this class of problems, two sets of auxiliary sources, given by line sources, i.e., the fundamental solutions of the 2D scalar Helmholtz equation, are needed [2]. The first set is that of the External Auxiliary Sources (EAS) located on S^{ext} (a fictitious contour conformal to S) which radiate in the absence of the scatterer and are such to reproduce the incident field in Ω^{int}. The second one is the set of the Internal Auxiliary Sources (IAS) located on S^{int}, which again radiate in free space and are such to reproduce the field radiated in Ω^{ext} by the current that the incident field induces in Ω. In these hypotheses, the incident and scattered fields can be approximately expressed as the following linear combinations, respectively:

$$\text{(a)}\quad E^z_{inc}(r) \cong \sum_{j=1}^{N_E} a_j H_0^{(2)}(k_0\,|\,r - r_j^E\,|)\,, \qquad \text{(b)}\quad E^z_{sc}(r) \cong \sum_{k=1}^{N_I} b_k H_0^{(2)}(k_0\,|\,r - r_k^I\,|) \tag{1}$$

wherein N_E and N_I are the number of EAS and IAS, a_j and b_k represent the excitation coefficients of the auxiliary line sources located in r_j^E and r_k^I, while r is the polar coordinate of the generic observation point on a measurement curve placed at a given distance from the physical boundary S.

Since the scattering matrix contains, by definition, the object's "response" to a collection of elementary inputs, these results can be exploited to achieve it. As a matter of fact, any generic input to the system S (i.e., inside Ω) can be expressed, through (1.a), as the weighted superposition of the fields radiated by N_E EAS located on S^{ext}, whereas, for a fixed EAS (acting as an incident field for S), the field scattered by S can be expressed, through (1.b), as the weighted superposition of the fields radiated by N_I IAS located on S^{int}. Therefore, we can synthetically express the overall field scattered by the system S under generic condition as

$$E_{sc}^{z}(r) \cong \sum_{j=1}^{N_E} a_j \sum_{k=1}^{N_I} s_{kj} H_0^{(2)}(k_0 \,|\, r - r_k^I \,|) \,, \tag{2}$$

wherein $\boldsymbol{a} = (a_1, ..., a_{N_E})$ represents the set of EAS coefficients required to reproduce the incident field actually impinging on the scatterer and, for a fixed j, the coefficients $\boldsymbol{s_j} = (s_{1j}, ..., s_{N_I j})$ represent the IAS excitations required to match the field scattered by S when illuminated by the j-th EAS with unitary excitation. From the comparison of (1.b) and (2), we can then write the matrix relationship $\boldsymbol{b} = \boldsymbol{S_g}\, \boldsymbol{a}$, wherein $\boldsymbol{S_g}$ denotes the $N_I \times N_E$ *generalized* scattering matrix (whose generic coefficient is s_{kj}) that completely characterizes the object's electromagnetic behavior. In order to determine the coefficients of $\boldsymbol{S_g}$, we must then evaluate, for each elementary EAS (with $a_j=1, \forall j$), the set of unknown internal sources which can approximate the corresponding actual scattered field. By denoting with $\boldsymbol{r} = (r_1, ..., r_P)$ the set of P collocation points on a measurement curve external to the object's boundary, this means that we have to solve N_E times the following linear system:

$$E_{sc\ j}^{z}(r_p) \cong \sum_{k=1}^{N_I} s_{kj} H_0^{(2)}(k_0 \,|\, r_p - r_k^I \,|) \qquad p=1,\ldots,P \tag{3}$$

where $j=1,...,N_E$ and r_k^I is the position of the k-th internal source, while $E_{sc\ j}^{z}$ is the field scattered by the object when illuminated by the j-th external unitary-amplitude source that is computed via some other numerical method.

Therefore, for a fixed j, the coefficients s_{kj} represent the j-th row of the G-SM $\boldsymbol{S_g}$ that, once calculated, allows us to evaluate, through eq. (3), the field scattered by the object under any incidence condition.

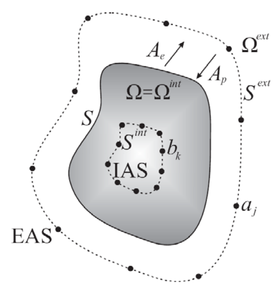

Figure 1. Generic 2D scatterer with MAS notations

3. Fixing the dimension of the G-SM

For the proposed generalized scattering matrix to be practically relevant, we need to assess its dimension, i.e., to fix the number (and possibly collocation) of the auxiliary sources which are required to reproduce, within a fixed accuracy, the actual fields. In this respect, MAS theory states that the superposition of auxiliary fields approaches the exact field when the number of auxiliary sources goes to infinity [2], whereas an effective numerical implementation would of course aim at keeping this number as low as possible. Despite this is an important topic in MAS literature (see, f.i., [3]), a largely redundant number of sources is usually adopted and only recently, by exploiting the properties of the fields radiated by bounded sources, quantitative criteria have been given to pick the *minimum* number of sources required to ensure a given accuracy when handling *smooth* metallic scatterers [4]. Moving from these latter results, in the following we devise a general procedure to fix the number of sources for a scatterer of arbitrary shape, assuming, for the sake of simplicity, equally spaced auxiliary sources.

Let us first determine the number of IAS needed to represent the scattered fields. These fields are actually radiated from the currents induced in the scatterer, so that they can be expressed, on a measurement curve (f.i., coinciding with the contour of the external sources S^{ext}), as the output of a linear integral operator:

$$A_e : J \in L^2(\Omega^{int}) \rightarrow \iint_{\Omega^{int}} G(r,r')J(r')dr' = E_{sc}(r) \in L^2(S^{ext}) \tag{4}$$

whose kernel $G(r,r')$ is the free space Green's function. From the properties of this function, we have the following two consequences (see [4] and references therein for details):

1. if the superposition of the fields generated by the auxiliary sources fits, with a given precision, the scattered field on the scatterer's surface S, it represents, by virtue of the uniqueness theorem, the scattered field at its exterior with the same (or higher) precision;
2. the set of the fields radiated by all finite families of point sources inside S is dense in the set of all fields on S.

From the above it descends that any scattered field can be approximated with arbitrary precision, on the scatterer's surface and at its exterior, by the superposition of the fields radiated by a *finite* number of auxiliary sources. Another important circumstance that descends from the properties of the kernel of A_e is the possibility of expressing the relationship between the source J and the scattered field E_{sc} in terms of the Singular Value Decomposition (SVD) [5]:

$$E_{sc}(r) = A_e(J) = \sum_{n=0}^{+\infty} \sigma_n \langle J(r'), u_n(r') \rangle v_n(r) \tag{5}$$

wherein $\langle \cdot, \cdot \rangle$ is the scalar product over Ω^{int}, u_n are the (right) singular functions which span the space of sources, v_n are the (left) singular functions which span the space of scattered fields and $\sigma_n > 0$ are the singular values. As a matter of fact, this decomposition expresses the generic scattered field as the combination of the "elementary" fields defined by the singular functions v_n. Due to the analytic properties of A_e such a set can be uniformly approximated by a finite rank one, with arbitrarily small error since the singular values go to zero as $n \to \infty$.

The above results suggest a way to determine the minimum number of IAS. As a matter of fact, for any given scatterer, after computing the discretized counterpart of A_e and its SVD, the behavior of the singular values identifies the index N at which truncating the summation (5) in such a way that, within the given accuracy, *any* scattered field outside S^{ext} is represented by a linear combination of the first N left singular functions. Then, a suitable choice for N_I would be such that the corresponding set of IAS can reproduce the N "elementary" scattered fields corresponding to $v_0,...,v_N$. Therefore, by denoting with r_k^I the position of the k-th auxiliary source on S^{int}, we have to pick the minimum value of N_I for which the difference

$$\| v_n(r) - \sum_{k=1}^{N_I} \beta_k H_0^{(2)}(k_0 \,| r - r_k^I |) \|^2 \tag{6}$$

is within the expected precision for all n=1,...,N. In (6), r spans the set of collocation points on S^{ext} for the scattered field. It is worth to note that, as the minimum number N of singular values increases when $S^{ext} \to S$ [6], evaluating the scattered field in the close proximity of the scatterer requires a larger set of conditions to be matched, thus possibly requiring a larger number of auxiliary sources. On the other hand, such an issue can be partially mitigated by properly positioning inside the scatterer the curve S^{int} wherein the IAS lie. A possible way to make this choice is to repeat the above described analysis for different positions of S^{int}.

As far as the incident field and the number of EAS are concerned, taking advantage of MAS theoretical foundations, it can be demonstrated that any field within Ω, i.e., any incident field, can be expressed as a linear combination of the fields radiated by the set of auxiliary sources placed on the contour S^{ext} with an infinitely small spacing. Hence, similar reasonings as above can be done by studying the properties of the integral operator, say A_p (non-conjugate transposed of A_e), which relates a source defined over the contour S^{ext} to the field radiated in the domain Ω. Therefore, by evaluating its SVD we can determine which is the set of the "elementary" incident fields (corresponding to the singular functions) that, for a given accuracy, have to be matched by an adequate number N_E of equally spaced EAS.

4. Numerical example

As an example of the above procedure, in the following we consider the problem of evaluating the field scattered from a non-canonical scatterer having a square cross-section of side 2λ (λ being the wavelength in free-space), see Fig. 2a, and the spatially variable-permittivity profile shown in Fig. 2b.

By numerically computing the singular system $\{u_n, \sigma_n, v_n\}$ of the discretized operator A_e as particularized to this geometry (assuming a square measurement curve of side $l_\Gamma = 3\lambda$), we get the normalized singular values shown in Fig. 2c. From this result, by fixing the accuracy at -32 dB (≈ 2%), it follows that $N = 23$ v_n left singular functions are the meaningful ones required to represent any scattered field outside Γ. On the other hand, as the singular values are the same for A_p and A_e [6], the same number of singular functions will be needed to span the set of all possible incident fields inside the object's domain. By evaluating the quantity (6), weighted by the relevant singular values, it results that 24 equally spaced sources (for both IAS and EAS) are required. In the same way, while still assuming N_I =24, we can notice that the error progressively reduces as long as the distance between S and S^{int} grows up to 0.3λ (i.e., l_S^{int}/λ=1.7). This latter is then assumed as the optimal value. It is worth to note that these results would hold true for any scatterer (metallic or dielectric) having the same cross-section as the one considered in this example.

By using the above number of auxiliary sources, we have solved eq. (3) to build the G-SM and used it to compute the scattered field for the case of a plane wave impinging from θ_{inc}=0 (with respect to the y axis). As it can be observed from Fig. 2d, the comparison with the result of a method-of-moment based forward solver confirms the expected accuracy of the proposed procedure.

5. Conclusions

A simple semi-analytical procedure to evaluate the generalized scattering matrix of a whatever complex-shaped 2D object has been presented. This procedure only requires a numerical pre-processing step to achieve the coefficients of a matrix, which can be afterwards directly exploited to determine the electromagnetic behavior of the structure at hand in any generic scattering problem.

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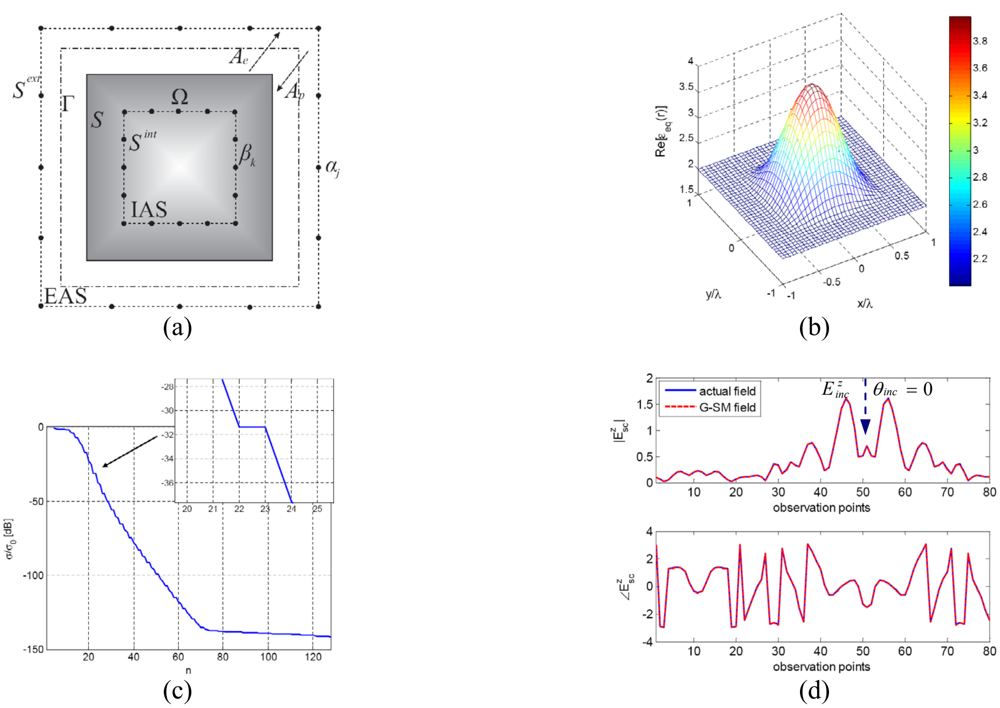

Figure 2. a) $2\lambda \times 2\lambda$ square scatterer with MAS notation; b) $Re[\varepsilon_{eq}(r)]$; c) Normalized σ_n for l_Γ/λ=3; d) Amplitude an phase of $E_{sc}^{z}(r)$ on Γ