Whispering Gallery Mode Elliptical Microring Laser-- Resonance and Field Computation

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Abstract

This paper introduces a computationally efficient method for determining the Whispering Gallery Mode (WGM) laser resonance and field distributions in an elliptical microring. An analytical method is applied to avoid computing high order Mathieu functions. Computed results show that the eccentricity of the ellipse affects the distribution of resonant frequencies and the laser field.

1. Introduction

The demonstration of lasers in luminescent conducting polymer thin films [1] has triggered studies of cylindrical microcavities formed with conductive polymers [2, 3]. It is reported that the lasing modes are of WGMs as ones studied by Wait [4]. Major studies on cylindrical-cavity lasers have been experimental [1, 2]. Simplified analysis for resonant modes was presented in [3] and Baktur et. al. [5] provided a more comprehensive theoretical description for WGM laser resonances in a circular microring. While microring geometry has been confined to a circular cross-section so far, we found that it is equally important to study the WGM laser in an elliptical microring because such structure has an extraordinary property when used as a pump source. It is the objective of this work to develop a computationally efficient treatment for determining the resonance and fields of a WGM laser in a microring with an elliptical cross-section.

2. Elliptical Microring Configuration

The configuration of the elliptical microring is as shown in Fig. 1, the major and minor axes of the inner boundary of the microring are denoted as a, b. For the outer boundary, the two axes are a' and b'. The thickness of the microring is d, and it is clear to see the following relations hold.

$$a' = a + d \ , \tag{1}$$

and

$$b' = b + d \, . \tag{1}$$

In elliptical coordinates, solutions for wave equations are combinations of Mathieu's functions [6]. Since WGMs are high order modes [4], high order Mathieu's functions have to be computed. Although it is possible to study the WGM resonance in an elliptical cavity by evaluating Mathieu functions [7], the mode numbers computed are limited due to the difficulty in convergence. Also, the formulation for calculating field components is rather tedious. Therefore, we approach the problem with a curve-fitting method to reduce the computational load.

It is helpful to work out relations between variables used in the coordinate system where the microring is studied. In Fig. 1, a point p on the outer ellipse can be located by any one pair of variables from (x, y), (l, ϕ) or (ξ_2, η). The relations between these three sets of variables are as follows:

$$x = l\cos\phi, y = l\sin\phi \, , \tag{2}$$

$$x = a'\cos\eta, y = b'\sin\eta, \tag{3}$$

$$l = \sqrt{x^2 + y^2}, \cos\phi = \pm\frac{a'}{l}\sqrt{\frac{l^2 - b'^2}{a'^2 - b'^2}}, \tag{4}$$

$$\cos\eta = \frac{l}{a'}\cos\phi, \sin\eta = \frac{l}{b'}\sin\phi, \tag{5}$$

and

$$\tanh\xi_2 = \frac{b'}{a'}. \tag{6}$$

The proposed method is to deduce the WGM field with propagation in a local osculating circle. At a point in the microring, the laser field is approximated by the field in the circle of curvature at that point. To begin the discussion, the outer ellipse (axes: *a'*, *b'*) is fit locally with circles of curvature as illustrated in Fig. 2. For example, at the point *P* on the ellipse (axes: *a'*, *b'*), the circle of curvature is C_1. C_1 is an osculating circle at *P*, and it has the same radius as the radius of curvature at the point *P*. The radius of curvature at a point (ξ_2, η), represented by $R(\eta)$, can be computed from the following formula [8]:

$$R(\eta) = (a'^2\sin^2\eta + b'^2\cos^2\eta)\exp(3/2)/(a'b'). \tag{7}$$

The WGM field at P_1 in the ellipse is viewed as having the same property as the WGM field at P_1 in the circle C_1. Therefore, computing the field at P_1 in C_1 approximates the field in the ellipse at P_1. Similarly, fields at P_2 and P_3 can be obtained by computing fields in the circle C_2 and C_3 at these points. When circles of curvature are fit into the ellipse at every point, then the field at any point inside (or outside, see P_2 in Fig. 2) the ellipse can be computed.

3. Whispering Gallery Mode Resonance

In the local circle (the circle of curvature), the circumferential propagation is contained in angular wave function, which is $e^{-j\nu\phi}$. In this wave function, ν is the angular wave number [5], and ϕ is the angular distance on the local circle. It is desirable to rewrite $e^{-j\nu\phi}$ into $e^{-j\nu(\eta)\phi(\eta)}$ to show local propagation.

When a WGM propagates along the ellipse, for a whole period, the increase in phase along the path of the angular propagation is $\int_{\eta=0}^{\eta=2\pi}\nu(\eta)\phi(\eta)d\eta$. In order to achieve a WGM resonance the phase increase needs to be an integer multiples of 2π when the electromagnetic wave finishes a whole period along the ellipse. So equation (9) holds for an integer *m*. $\nu(\eta)$ is closely related to the radius of the curvature [5], and is the order of the Bessel functions that describe the WGM in the local microring at η.

$$\int_{\eta=0}^{\eta=2\pi}\nu(\eta)\phi(\eta)d\eta = 2m\pi. \tag{8}$$

$\phi(\eta)$ is the angular distance along the local circle at η and the corresponding length on the local circle is

$$l(\eta) = R(\eta)\phi(\eta). \tag{9}$$

It should be noted that at the vicinity of (ξ, η), $l(\eta)$ can be approximated by the arc length of the ellipse along $d\eta$ and yields

$$l(\eta) = \sqrt{a'^2\sin^2\eta + b'^2\cos^2\eta}\, d\eta. \tag{10}$$

Using (7), (9) and (10), (8) can be re-written into

$$\int_0^{2\pi}\frac{a'b'}{a'^2\sin^2\eta + b'^2\cos^2\eta}\nu(\eta)d\eta = 2m\pi. \tag{11}$$

When the ellipse takes the limit to a circle, (11) gives ν=*m*. This result is the same one as discussed in [5] for a circular microring resonance. By using a zero finding routine, $\nu(\eta)$ can be computed from the characteristic equation of the local microring. WGM resonances for transverse electric and magnetic modes (transverse to the axis of the elliptical cylinder) are computed for elliptical dielectric cylinders with same perimeters and varied axial ratios. We found that the axial ratio of the ellipse affects the resonance by shifting the resonant wavelength. The spacing between the resonant wavelengths does not change significantly according to the shape of the ellipse as long as the perimeter of the ellipse stays the same.

4. WGM Laser Field

When an excitation is placed near P_0 in the osculating circle C_0 (Fig. 3), the WGM field at P_0 can be determined from the radius of C_0 and v_0, which is the order of the WGM. The field at P_1, which is located next to P_0, is on the osculating circle C_1. When P_0 and P_1 are in the vicinity of each other, both of them can be approximately viewed as in the circle C_1, and accordingly satisfy the following:

$$F(P_1) = F(P_0)e^{j\nu_1\Delta\phi} = F(P_0)\exp(j\nu_1\Delta\phi / R_1) = F(P_0)\exp(j\nu_1\Delta l / R_1). \tag{12}$$

The arc length can be computed from

$$\Delta l = \sqrt{a^2 \sin^2 \eta_1 + b^2 \cos^2 \eta_1}\, \Delta\eta , \tag{13}$$

where $\Delta\eta$ is the angular variation from P_0 to P_1 along the ellipse e, η_1 is the angular coordinate of P_1 on the ellipse e and the relation between η and ϕ is given by (5) .

By using (13) we have

$$j\nu_1\Delta\phi = jab\nu_1 /(a^2 \sin^2 \eta_1 + b^2 \cos^2 \eta_1)\Delta\eta . \tag{14}$$

Let $\Gamma_1 = ab\nu_1 /(a^2 \sin^2 \eta_1 + b^2 \cos^2 \eta_1)$, and (13) becomes

$$F(P_1) = F(P_0)e^{j\Gamma_1\Delta\eta} . \tag{15}$$

Similarly, fields at P_2 and P_3 can be computed from

$$F(\mathrm{P}_2) = F(P_1)e^{j\Gamma_2\Delta\eta} = F(P_0)e^{j\Gamma_1\Delta\eta}e^{j\Gamma_2\Delta\eta} = F(P_0)e^{j(\Gamma_1+\Gamma_2)\Delta\eta} . \tag{16}$$

and

$$F(P_3) = F(P_0)e^{j(\Gamma_1+\Gamma_2+\Gamma_3)\Delta\eta} . \tag{17}$$

Iterating this process gives the field at P_N to be

$$F(\eta) = F(P_0)\exp(j\sum_{i=1}^{N} \Gamma_i\Delta\eta) . \tag{18}$$

When N approaches infinity, (18) becomes

$$F(\eta) = F(P_0)\exp(j\int_{\eta_0}^{\eta} \Gamma(\eta)d\eta) . \tag{19}$$

In order to have (19) valid for every η, we define Γ=0 at η_0, and for anywhere else we have

$$\Gamma(\eta) = ab\nu(\eta) /(a^2 \sin^2 \eta + b^2 \cos^2 \eta) . \tag{20}$$

For a point P_η^r, we can find its projection (ξ_0, η) on the ellipse e. The distance from P_η^r to the ellipse is d_r. Then, on the ellipse with two axes (a+d_r, b+d_r), we locate a point P_0^r that can also be defined by the osculating circle C_0 and the distance d_r. For example, in Fig. 3, P_0^r is at (R_0+d, π/2), R_0 is the radius of C_0. The field at P_0^r can be easily computed and then the field at P_η^r can be determined from

$$F(\mathrm{P}_\eta^\mathrm{r}) = F(\mathrm{P}_0^\mathrm{r})\exp(j\int_{\eta_0}^{\eta} \Gamma(\eta)d\eta) . \tag{21}$$

It is important to make sure the correct η is used. In (21), η is associated with the outer ellipses e, and it can be determined from (3) or (5). With this curve-fitting method, WGM field excited by a given excitation in a microring can be computed.

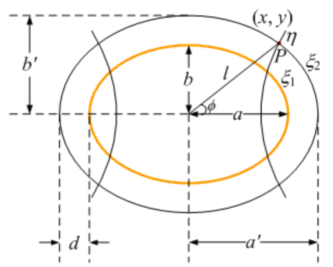

Fig. 1 Configuration of elliptical microring (cross-section view).

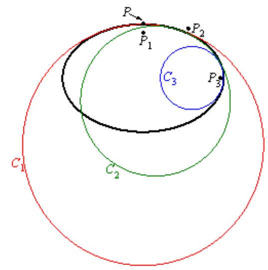

Fig. 1. Ellipse with its osculating circles at three points.

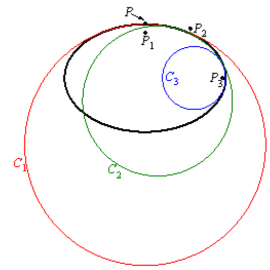

Fig. 2. Illustration of the elliptical microring with osculating circles.

5. Conclusion

The paper introduces a method to compute WGM resonance and field components in an elliptical microring. The method assumes that the WGM field at a point in the elliptical microring is of the same nature as the WGM field in the circle of the curvature of the ellipse at that point. This assumption is valid for an electronically large microring (the circumference of the microring is 20 times larger than the wavelength) with elliptical cross-section. We found that a change in the axial ratio of the elliptical microring results in a shift in the WGM resonant frequencies. One needs to pay attention to the thickness of the microring. As discussed in [5], the thickness of the ring to support the WGM is associated with the dimension of the structure. The bigger the radius of the microring, the thicker the microring must be. This means that for an elliptical microring, it has to be thick enough to support WGM at two ends of the minor axis (For example, P_0 in Fig. 3) for the osculating circle has the largest radius at this point. On the other hand, when a relatively thick circular microring is deformed into an elliptical microring, there may shape a sharp edge at the major axis at the inner boundary and it may affect the computation and validity of the method presented.

7. References

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