Analysis of the Light Propagation in a Disordered Waveguide System

Akira Komiyama
Osaka Electro-Communication University
18-8 Hatsu-cho Neyagawa-shi 572-8530,Japan
Tel:+81-72-824-1131, Fax:+81-72-824-0014, E-mail:komiyama@isc.osakac.ac.jp

Abstract

The propagation of light in a disordered waveguide system composed of randomly different cores in size is treated as an initial value problem of the coupled mode equations and the average amplitude of light is theoretically derived using the scattering matrix. The damping factor obtained is compared with the numerical results. Both results are in good agreement with each other.

1. Introduction

Wave functions for electrons in a disordered atomic system are localized and are concentrated into a narrow region of space[1]. The density of states for an electronic system represents the energy distribution of electron and can be obtained from the Green's function. One of the methods to obtain the Green's function for a disordered system is the coherent potential approximation[2]. The approximation is a starting point of the diagram theory for localization of electron[3].

In a disordered waveguide system composed of randomly different cores in size mode waves are also localized and are concentrated into a narrow region of several cores[4]. The modal density of a waveguide system can also be obtained by using the coherent potential approximation[5]. In this approximation the scattering matrix plays an important role. When one of core is illuminated at the input end of the system a number of localized modes are excited and their modes propagate. The propagation properties have been clarified by solving numerically the coupled mode equations[6]. So far, the theoretical discussion, however, has not been reported.

In this paper the propagation of light in a disordered waveguide system is treated as an initial value problem of the coupled mode equations and the average amplitude of light is theoretically derived using the scattering matrix. The damping factor obtained is compared with the numerical results. Both results are in good agreement with each other.

2. Ordered Waveguide System

For an ordered waveguide system composed of identical cores the following coupled modes equations can be written,

$$\frac{dc_n}{dz} = -j\beta c_n - j\kappa(c_{n-1} + c_{n+1}) \tag{1}$$

c_n is the amplitude of the mode in the n-th core and β is the propagation constant of the mode. κ is the mode coupling coefficient between cores. A slowly varying amplitude a_n is introduced,

$$c_n = a_n e^{-j\beta z} \tag{2}$$

Then we have

$$\frac{da_n}{dz} = -j\kappa(a_{n-1} + a_{n+1}) \tag{3}$$

The solution $G_0(n, l, z)$ to Eq.(3) with an initial condition $a_n(0) = \delta_{nl}$ where δ_{nl} is the Kronecker delta symbol is given by

$$G_0(n, l, z) = \frac{J_{n-l}(2\kappa z)}{j^{n-l}} \tag{4}$$

where J_n is the Bessel function of order n.

The Laplace transform of Eq.(3) with an initial condition $a_n(0) = \delta_{n0}$ is

$$a\tilde{a}_n - \delta_{n0} = -j\kappa(\tilde{a}_{n-1} + \tilde{a}_{n+1}) \tag{5}$$

We rewrite Eq.(5) in a matrix form,

$$\tilde{\boldsymbol{M}}\tilde{\boldsymbol{a}} = \boldsymbol{a(0)} \tag{6}$$

where $(\boldsymbol{a(0)})_n = \delta_{n0}$ and

$$(\tilde{\boldsymbol{M}})_{nm} = \begin{cases} s, & m = n \\ j\kappa, & m = n-1,\ n+1 \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

$\tilde{\boldsymbol{a}}$ is given by

$$\tilde{\boldsymbol{a}} = \tilde{\boldsymbol{M}}^{-1}\boldsymbol{a(0)} \tag{8}$$

where

$$\begin{aligned} (\tilde{\boldsymbol{M}}^{-1})_{nm} &= (\tilde{\boldsymbol{G}}_0)_{nm} \\ &= \tilde{G}_0(n, m, s) \\ &= \frac{(\sqrt{s^2 + 4\kappa^2} - s)^{|n-m|}}{(j2\kappa)^{|n-m|}\sqrt{s^2 + 4\kappa^2}} \end{aligned} \tag{9}$$

$\tilde{G}_0(n, m, s)$ is the Laplace transform of $G_0(n, m, z)$. The light propagation in an ordered system can be completely described in terms of $\tilde{\boldsymbol{G}}_0$.

3. Scattering from an Imperfection Core

For a waveguide system with a single imperfection core the coupled mode equation is

$$\tilde{\boldsymbol{M}}\tilde{\boldsymbol{a}} = \boldsymbol{a(0)} - j\boldsymbol{\Delta\beta}_l\tilde{\boldsymbol{a}} \tag{10}$$

The matrix $\boldsymbol{\Delta\beta}_l$ is

$$(\boldsymbol{\Delta\beta}_l)_{nm} = \begin{cases} \delta\beta_l, & n = m = l \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

where $\delta\beta_l$ is the fluctuation of the propagation constant of the l-th core. Multiplying Eq.(10) by the inverse matrix $\tilde{\boldsymbol{M}}^{-1}$ ($=\tilde{\boldsymbol{G}}_0$) we have

$$\tilde{\boldsymbol{a}} = \tilde{\boldsymbol{G}}_0\boldsymbol{a(0)} - j\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l\tilde{\boldsymbol{a}} \tag{12}$$

An alternative expression can be obtained by iterating on $\tilde{\boldsymbol{a}}$:

$$\begin{aligned} \tilde{\boldsymbol{a}} &= \tilde{\boldsymbol{G}}_0\boldsymbol{a(0)} - j\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l\tilde{\boldsymbol{G}}_0\boldsymbol{a(0)} - \tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l\tilde{\boldsymbol{G}}_0\boldsymbol{a(0)} + \cdots \\ &= \tilde{\boldsymbol{G}}_0\boldsymbol{a(0)} - j\tilde{\boldsymbol{t}}_l\tilde{\boldsymbol{G}}_0\boldsymbol{a(0)} \end{aligned} \tag{13}$$

where $\tilde{\boldsymbol{t}}_l$ is the scattering matrix for a single imperfection core,

$$\begin{aligned} \tilde{\boldsymbol{t}}_l &= \tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l - j\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l\tilde{\boldsymbol{t}}_l \\ &= \tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l - j\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l - \tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}_l + \cdots \end{aligned} \tag{14}$$

$\tilde{\boldsymbol{t}}_l$ can be exactly calculated, which is given by

$$(\tilde{\boldsymbol{t}}_l)_{nm} = \frac{\delta\beta_l(\tilde{\boldsymbol{G}}_0)_{nl}}{1 + j\delta\beta_l(\tilde{\boldsymbol{G}}_0)_{ll}}\delta_{ml} \tag{15}$$

4. Disordered Waveguide System

The coupled mode equation for a disordered system is

$$\tilde{\boldsymbol{M}}\tilde{\boldsymbol{a}} = \boldsymbol{a}(\boldsymbol{0}) - j\boldsymbol{\Delta\beta}\tilde{\boldsymbol{a}} \tag{16}$$

$\boldsymbol{\Delta\beta}$ is a diagonal matrix,

$$(\boldsymbol{\Delta\beta})_{nm} = \begin{cases} \delta\beta_m, & n = m \\ 0, & n \neq m \end{cases} \tag{17}$$

Then we have

$$\begin{aligned} \tilde{\boldsymbol{a}} &= \tilde{\boldsymbol{G}}_0\boldsymbol{a}(\boldsymbol{0}) - j\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}\tilde{\boldsymbol{a}} \\ &= \tilde{\boldsymbol{G}}_0\boldsymbol{a}(\boldsymbol{0}) - j\tilde{\boldsymbol{T}}\tilde{\boldsymbol{G}}_0\boldsymbol{a}(\boldsymbol{0}) \end{aligned} \tag{18}$$

$\tilde{\boldsymbol{T}}$ is the scattering matrix for the total system,

$$\tilde{\boldsymbol{T}} = \tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta} - j\tilde{\boldsymbol{G}}_0\boldsymbol{\Delta\beta}\tilde{\boldsymbol{T}} \tag{19}$$

$\tilde{\boldsymbol{T}}$ and $\boldsymbol{\Delta\beta}$ are written as

$$\tilde{\boldsymbol{T}} = \sum_l \tilde{\boldsymbol{T}}_l, \quad \boldsymbol{\Delta\beta} = \sum_l \boldsymbol{\Delta\beta}_l \tag{20}$$

$\tilde{\boldsymbol{T}}_l$ can be expressed in terms of $\tilde{\boldsymbol{t}}_l$,

$$\begin{aligned} \tilde{\boldsymbol{T}}_l &= \tilde{\boldsymbol{t}}_l - j\sum_{p\neq l}\tilde{\boldsymbol{t}}_l\tilde{\boldsymbol{T}}_p \\ &= \tilde{\boldsymbol{t}}_l - j\sum_{p\neq l}\tilde{\boldsymbol{t}}_l\tilde{\boldsymbol{t}}_p - \sum_{p\neq l}\sum_{q\neq p}\tilde{\boldsymbol{t}}_l\tilde{\boldsymbol{t}}_p\tilde{\boldsymbol{t}}_q - \cdots \end{aligned} \tag{21}$$

The ensemble average of $\tilde{\boldsymbol{a}}$ is

$$\begin{aligned} \langle\tilde{\boldsymbol{a}}\rangle &= \tilde{\boldsymbol{G}}_0\boldsymbol{a}(\boldsymbol{0}) - j\langle\tilde{\boldsymbol{T}}\rangle\tilde{\boldsymbol{G}}_0\boldsymbol{a}(\boldsymbol{0}) \\ &\sim \tilde{\boldsymbol{G}}_0\boldsymbol{a}(\boldsymbol{0}) - j\sum_l\langle\tilde{\boldsymbol{t}}_l\rangle\tilde{\boldsymbol{G}}_0\boldsymbol{a}(\boldsymbol{0}) \end{aligned} \tag{22}$$

Here the overall $\tilde{\boldsymbol{T}}$ is approximated by the sum of single scatterings from individual cores.

$$\tilde{\boldsymbol{T}} \sim \sum_l \tilde{\boldsymbol{t}}_l \tag{23}$$

Then the amplitude of light in the illuminated core $\langle\tilde{a}_0\rangle$ is

$$\begin{aligned} \langle\tilde{a}_0\rangle &\sim (\tilde{\boldsymbol{G}}_0)_{00} - j\sum_l\left\langle\frac{\delta\beta_l(\tilde{\boldsymbol{G}}_0)_{0l}}{1 + j\delta\beta_l(\tilde{\boldsymbol{G}}_0)_{ll}}\right\rangle(\tilde{\boldsymbol{G}}_0)_{l0} \\ &\sim (\tilde{\boldsymbol{G}}_0)_{00} - \delta\beta^2\sum_l(\tilde{\boldsymbol{G}}_0)_{0l}(\tilde{\boldsymbol{G}}_0)_{ll}(\tilde{\boldsymbol{G}}_0)_{l0} \\ &\sim \frac{1}{\sqrt{s^2+4\kappa^2}} - \frac{\delta\beta^2}{(s^2+4\kappa^2)^{\frac{3}{2}}}\left\{\frac{s}{2\kappa} - \frac{1}{2}\left(\frac{s}{2\kappa}\right)^3 + \cdots\right\} \end{aligned} \tag{24}$$

where $\delta\beta^2$ is the variance of the fluctuation of the propagation constants. Assuming $\delta\beta$ to be small the ensemble average is approximated by the first term of the series expansion. By taking the inverse Laplace transform of Eq.(24) we have

$$\begin{aligned} \langle a_0(z)\rangle &\sim J_0(2\kappa z) - \frac{\delta\beta^2}{2\kappa}zJ_0(2\kappa z) + \cdots \\ &\sim J_0(2\kappa z)e^{-\alpha z/2} \end{aligned} \tag{25}$$

where α is the damping factor,

$$\alpha = \frac{\delta\beta^2}{\kappa} \tag{26}$$

The dependence of the damping factor on $\delta\beta$ is shown in Fig.1. A solid line is the theoretical result(26) and dots are the results obtained by solving numerically the coupled mode equations. A value of the mode coupling coefficient κ is $0.868(1/mm)$. Both results are in good agreement with each other.

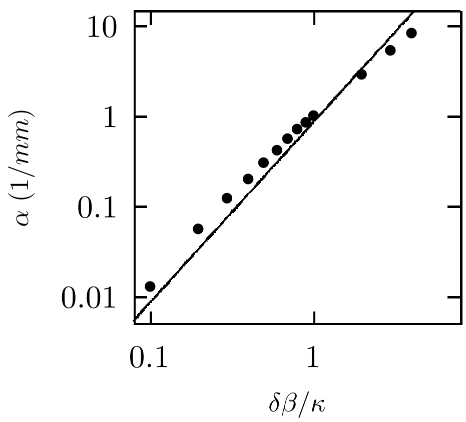

Fig.1 Dependence of the damping factor on $\delta\beta$. A solid line is the theoretical result and dots are the numerical results.

5. Conclusions

The propagation of light in a disordered waveguide system has been treated as an initial value problem of the coupled mode equations and the average amplitude of light has been theoretically derived by approximating the overall scattering matrix by the sum of single scatterings from individual cores. The average amplitude decreases exponentially with increasing distance. The damping factor obtained is in good agreement with the numerical results.

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