Computation of the One-Dimensional Free-Space Periodic Green's Function for Leaky Waves Using the Ewald Method

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Abstract

The analysis of one-dimensional (1D) periodic leaky-wave antennas in free space using the method of moments requires the 1D free-space periodic Green's function (FSPGF) for a 1D array of point sources. Among various techniques to accelerate the computation of the FSPGF is the Ewald method [1]. The application and validity of the Ewald method for evaluating the 1D FSPGF for complex wavenumbers have recently been reported by the authors [2]. In this presentation, the computational efficiency of the FSPGF for complex wavenumbers using the Ewald method is investigated and compared with the spectral series representation of the FSPGF.

1. Ewald Method for Leaky Waves

The FSPGF for a periodic (along z) infinite linear array of point sources with period d and phasing wavenumber k_{z0} (see Fig. 1) is given by a spatial sum of spherical waves as

$$G(\boldsymbol{r},\boldsymbol{r}') = \sum_{n=-\infty}^{\infty} e^{-jk_{z0}nd} \frac{e^{-jkR_n}}{4\pi R_n} . \tag{1}$$

This spatial series is extremely slowly convergent for real wavenumbers k_{z0} and is divergent (as n tends to negative infinity) for complex wavenumbers corresponding to wave attenuation in the positive z direction, as encountered for a leaky wave. An alternative series representation of the Green's function is the spectral series representation

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{d} \sum_{q=-\infty}^{\infty} e^{-jk_{zq}z} \frac{H_0^{(2)}(k_{\rho q}\rho)}{4j}, \tag{2}$$

where

$$k_{\rho q} = \sqrt{k^2 - k_{zq}^2} \qquad \text{and} \qquad k_{zq} = k_{z0} + \frac{2\pi q}{d},$$

which converges for both real and complex waves, though the convergence becomes slower as the radial distance ρ from the z axis tends to zero (it does not converge for $\rho = 0$).

In [3], the Ewald method was successfully employed to evaluate the FSPGF for real wavenumbers k_{z0}. However, for complex wavenumbers $k_{z0} = \beta - j\alpha$ (corresponding to a leaky wave), a new investigation is warranted, since it is not clear for what regions of the complex $k_{\rho q}$ plane the fundamental Ewald identity

$$\frac{H_0^{(2)}(k_{\rho q}\rho)}{4j} \equiv \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-\rho^2 s^2 + \frac{k_{\rho q}^2}{4s^2}}}{s} ds \tag{3}$$

is valid, especially since the Hankel function in the above equation is not analytic and has a branch cut along the negative real axis. In [2] we investigated this issue in order to arrive at an extension of the Ewald method that is valid for complex waves. In this paper we summarize this extension and also investigate the numerical computation of the FSPGF using the Ewald method versus the spectral series in Eq. (2).

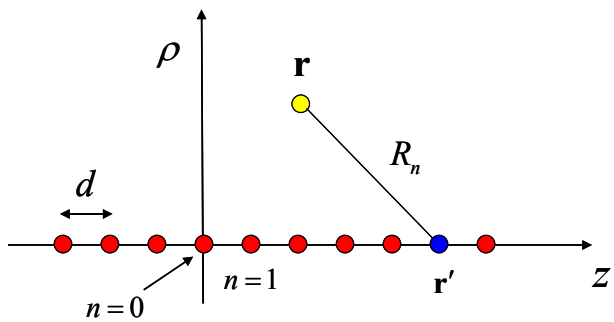

Fig. 1. A periodic linear array of point sources with period d. R_n is the distance between the observation point $\boldsymbol{r}$ and the nth source point $\boldsymbol{r}'$.

In the Ewald method the FSPGF is expressed as the sum of an "Ewald spectral" and an "Ewald spatial" series as follows:

$$G(\boldsymbol{r},\boldsymbol{r}') = G^E_{spectral}(\boldsymbol{r},\boldsymbol{r}') + G^E_{spatial}(\boldsymbol{r},\boldsymbol{r}') \tag{4}$$

$$G^E_{spectral}(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2\pi d}\sum_{q=-\infty}^{\infty} e^{-jk_{zq}z}\int_0^E \frac{e^{-\rho^2 s^2 + \frac{k_{\rho q}^2}{4s^2}}}{s}\,ds \tag{5}$$

$$G^E_{spatial}(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2\pi d}\sum_{q=-\infty}^{\infty} e^{-jk_{zq}z}\int_E^\infty \frac{e^{-\rho^2 s^2 + \frac{k_{\rho q}^2}{4s^2}}}{s}\,ds\ , \tag{6}$$

where $k_{\rho q}$ and k_{zq} are defined the same as in the spectral method, and E is the Ewald splitting parameter, typically chosen as $E = \sqrt{\pi}/d$ in order to achieve optimum convergence [3].

Following the mathematical transformations outlined in [3] for the case of complex $k_{\rho q}$, Eq. (5) can be expressed in a final form as

$$G^E_{spectral}(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi d}\sum_{q=-\infty}^{\infty} e^{-jk_{zq}z}\sum_{p=0}^{\infty}(-1)^p\frac{(\rho E)^{2p}}{p!}E^G_{p+1}\left(-\frac{k_{\rho q}^2}{4E^2}\right), \tag{7}$$

where $E^G_p(z)$ is the pth-order generalized exponential integral, defined by analytic continuation of the usual pth-order exponential integral function $E_p(z)$. In particular,

$$E^G_1(z) = E_1(z) + j2\pi m\,, \tag{8}$$

where

$$E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t} dt \,, \quad -\pi \le \arg z \le \pi \,, \tag{9}$$

is the usual first-order exponential integral function. The generalized exponential integral satisfies the same recurrence relation as does the usual exponential integral, namely

$$E_{p+1}^G(z) = \frac{1}{p}\left(e^{-z} - z\, E_p^G(z)\right),\ p = 1,2,3,\ldots\,. \tag{10}$$

This maybe used to calculate all of the needed orders p from $E_1(z)$.

The introduction of the generalized exponential integral is important for complex values of $k_{\rho q}$ to ensure that the correct path of integration in Eq. (9) is used for any complex value of $k_{\rho q}$. In particular, in the definition of the usual $E_1(z)$ function, the path stays in either the upper or lower half-plane, according to the location of the argument z, and does not cross the real axis. However, for a complex value of $k_{\rho q}$, the path may need to cross the negative real axis, in which a residue contribution is added from the pole at the origin. The choice of the integer m in Eq. (8) depends on the phase angle of $k_{\rho q}$ as

$$\begin{cases} -\pi < \mathrm{Arg}\left(k_{\rho q}\right) < 0\,, & m = 0 \\ 0 < \mathrm{Arg}\left(k_{\rho q}\right) < \pi\,, & m = -1\,. \end{cases} \tag{11}$$

Equation (6) requires no modifications for complex wavenumbers, and can be expressed as

$$G_{spatial}^E(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{8\pi}\sum_{n=-\infty}^{\infty} \frac{e^{-jk_{z0}nd}}{R_n}\left[e^{jkR_n}\,\mathrm{erfc}\,(R_n E + j\frac{k}{2E}) + e^{-jkR_n}\,\mathrm{erfc}\,(R_n E - j\frac{k}{2E})\right] \tag{12}$$

where $R_n = \sqrt{\rho^2 + \left(z - nd\right)^2}$ and $\mathrm{erfc}\,(z)$ is the complementary error function.

2. Computational Efficiency

In this section, we compare the computational efficiency of the FSPGF evaluated using Eq. (2), the spectral series representation, versus the Ewald representation of the FSPGF evaluated using Eq. (4). Figure 2 shows the computational efficiency of the two FSPGF representations considered, viz., the Ewald method and the spectral series for a typical leaky wave number $k_{z0} = (3.0 - j\,0.02)\,k_0$ corresponding to a slow-wave structure that radiates via the q = -1 space harmonic (Floquet wave). The period is chosen as $d = 0.4\lambda_0$. In each method the FSPGF is calculated to an accuracy of at least 10 significant figures. Results are shown for both z = 0 and z = d/2. The two plots (almost identical) verify that the conclusions are not sensitive to the value of z. The Ewald method is seen to be more efficient for radial distances less than about one quarter of a wavelength.

3. Conclusions

The Ewald method, which is a very efficient method for calculating the 1D free-space periodic Green's function, has been extended to allow for complex values of the phase wavenumber k_{z0}. This enables the

method to be applied to leaky waves, so that periodic leaky-wave antenna structures may be analyzed using the Ewald method. This extension required the modification of the paths of integration used in the complex plane to calculate the exponential integral function that appears in the method. A generalized exponential integral function was thus introduced that is the analytic continuation of the usual exponential integral function. With this modification, the Ewald formulation given in [3] for real wavenumbers becomes valid for complex wavenumbers.

The computational efficiency of the Ewald method was investigated and compared with the spectral series representation of the free-space periodic Green's function. It was observed that the Ewald method is extremely efficient for smaller values of radial distance ρ from the z axis. However, for moderate values of radial distance (typically around a quarter wavelength) the spectral series becomes more efficient, as it requires only a few terms to converge.

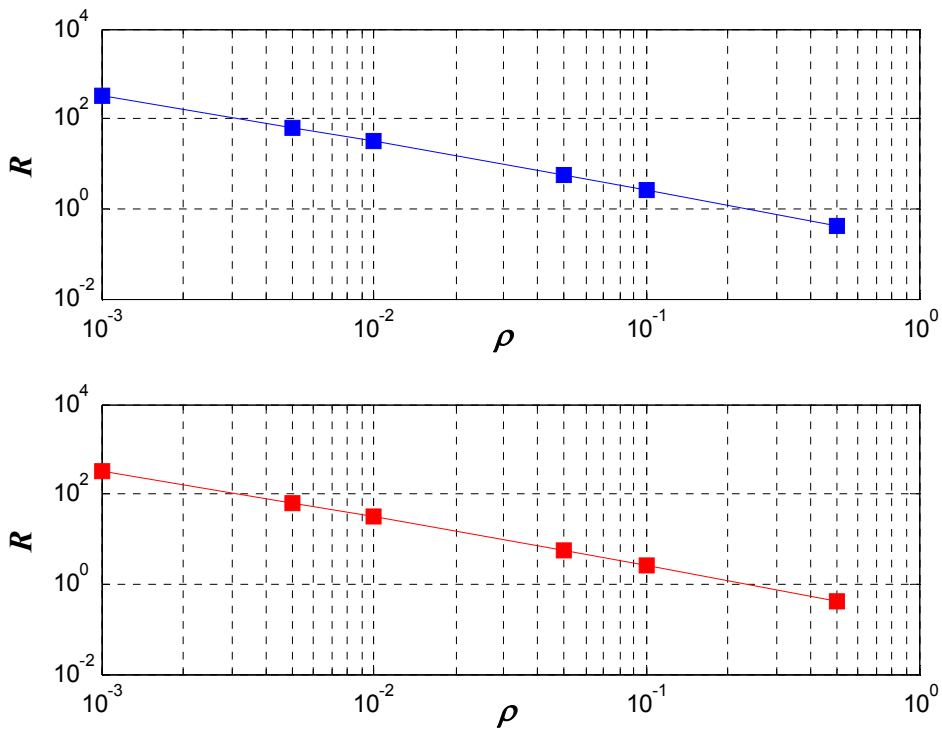

Fig. 2. Computational efficiency of the Ewald method vs. the spectral method. The cost ratio R represents the average time needed to calculate the Green's function using the spectral method relative to that needed for the Ewald method. Results are shown for two different values of z. The top plot is for $z = 0$, and the bottom plot is for $z = d/2$. (The two plots are nearly identical.)

4. References

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