Interpolation of 2D Layered-Medium Periodic Green's Function

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Abstract

We investigate various ways to use interpolation to speed up the calculation of the periodic 2D Green's function in layered media. Interpolation in both the spatial and spectral domains is explored. In the spatial domain, interpolation is applied directly to the periodic Green's function after it is regularized (smoothed and accelerated) by extracting periodic homogenous-medium periodic Green's functions. In the spectral domain, interpolation is applied to the voltage and current functions that comprise the spectral-domain Green's function. Sinusoidal interpolation may be used to calculate these quantities exactly.

1. Introduction

An important problem in many applications is the efficient computation of the two-dimensional (2D) periodic Green's functions in layered media. This corresponds to the calculation of the potential from a 2D infinite periodic set of lattice points arranged in the *xy* plane, with *z* being the direction normal to the layers. The lattice vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ in the plane of the sources may be arbitrary, as shown in Fig. 1.

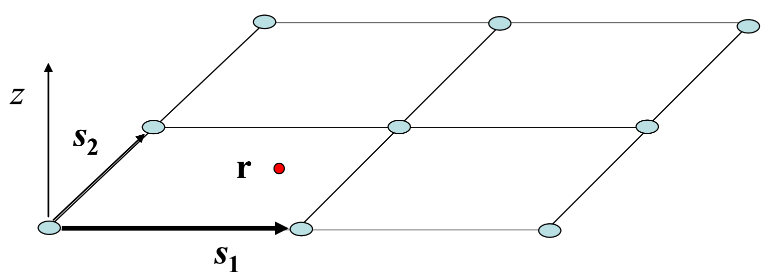

Figure 1. Geometry of an infinite 2D lattice of source points on a general skewed lattice in the *xy* plane, residing within a layered medium (layers not shown). An observation point within the (0,0) unit cell at an arbitrary *z* is also shown.

In layered media, the two-dimensional periodic Green's function has a spectral representation of the form

$$G(\mathbf{r},\mathbf{r}') = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \tilde{G}_{pq}(\mathbf{r},\mathbf{r}'), \tag{1}$$

where the sum is over spectral components $\tilde{G}_{pq}$. When the source and observation points are in the same or adjacent layers, asymptotic forms of $\tilde{G}_{pq}$ are typically removed from the Green's function to regularize it and accelerate the computation. The removed terms represent direct and quasi-image contributions (i.e., first bounce reflections from interfaces adjacent to the source layer). The removed terms, corresponding to 2D periodic Green's functions for homogeneous media, are added back in a form that is itself accelerated using the Ewald method [1–4]. The Ewald method uses a representation for the homogeneous medium periodic Green's function that is a sum of a "modified spectral" and "modified spatial" series. Each one is a rapidly converging sum, exhibiting Gaussian convergence. The resulting Green's function representation may be written as

$$G(\mathbf{r},\mathbf{r}') = G^{reg}(\mathbf{r},\mathbf{r}') + G^{\text{Ewald}}_{\text{spatial}}(\mathbf{r},\mathbf{r}') + G^{\text{Ewald}}_{\text{spectral}}(\mathbf{r},\mathbf{r}'). \tag{2}$$

The regularized Green's function is non-singular at the lattice source points when the lattice is embedded within a layer; when the lattice is at an interface there still exists a low-order singularity, but at least the potentials are finite at the lattice source points. In either case, the regularized periodic Green's function is much smoother and therefore more amenable to interpolation than the "unregularized" one. Furthermore, the regularized Green's function can be computed much faster than the "unregularized" one, since the subtraction of the homogenous-medium terms accelerates the convergence

Here we propose two approaches for interpolation of the regularized Green's function. The first one is a direct interpolation technique, which requires a four-dimensional interpolation in the space domain, namely, in the variables ($x-x'$, $y-y'$, z and z'). The alternative method is to tabulate and interpolate, as a function of the transverse wavenumber k_ρ, the transmission line voltage, current, and interface reflection coefficient functions that appear in the layered-medium Green's function. This requires only a one-dimensional interpolation, but the Green's function series must then be computed for each observation and source point.

To further enhance the speed-up, a three-dimensional interpolation is applied to the extracted parts (homogeneous periodic Green's functions, which are evaluated by the Ewald method) after they are regularized by extracting contributions from the four corner source terms in the (0,0) unit cell.

2. Interpolation techniques for the regularized Green's function

A) Interpolation in the space domain (4D interpolation)

Using the periodic property of the regularized Green's function, we need only interpolate observation points within a single unit cell, assumed to be the (0,0) unit cell. For observation points outside the unit cell, the observation point is shifted an integral number of lattice periods into the unit cell, with the result multiplied by a progressive phase shift term corresponding to the number of lattice periods translated in each dimension. Since even the "regularized" Green's function components typically have slope discontinuities across the interfaces, one should not interpolate across media interfaces. Furthermore, different sampling densities may also be needed in different layers. Hence, a separate set of tables should be maintained for every combination of source point and observation point layers.

Though a straightforward interpolation of $G^{reg}(\mathbf{r},\mathbf{r}')$ yields a fast interpolation, such a direct approach has some disadvantages. One is the number of sample points required: a simple, three-point (quadratic) interpolation in the four variables ($x-x'$, $y-y'$, z and z'), for example, requires $3^4 = 81$ interpolation points. By using this method, a significant speed-up factor is gained (initial tests show a speed-up factor on the order of 10). By using only interpolation points in a 4-D simplex, it should be possible to reduce the number of interpolation points to as low as 15, resulting in a further speed-up.

B) Interpolation in the spectral domain (1D interpolation)

For a given transverse wavenumber, the components of the spectral transverse electric and magnetic fields in a horizontal plane may be interpreted as voltages and currents on a transmission-line analog (referred to as the *transverse equivalent network* or TEN) of the medium. The transmission line network models propagation and reflection of TM_z and TE_z plane waves in the z direction (normal to the interfaces).

The TE_z and TM_z waves each have their own TEN, with different characteristic impedances in the network for each of the two types of waves in each layer. The two transmission line models for the TE_z and TM_z waves are used to construct the total Green's functions.

The Green's function has the generic form

$$G(\rho-\rho';z,z') = \sum_{\alpha=\mathrm{TE,TM}} \sum_m \sum_n \left(g_{vi}^\alpha V_i^\alpha + g_{ii}^\alpha I_i^\alpha + g_{vv}^\alpha V_v^\alpha + g_{iv}^\alpha I_v^\alpha \right) \tag{3}$$

where

$V_i^\alpha = V_i^\alpha\left(k_{\rho mn}, z, z'\right)$ is the voltage at z due to a unit parallel current source at z',

$V_v^\alpha = V_v^\alpha(k_{\rho mn}, z, z')$ is the voltage at z due to a unit series voltage source at z',

$I_i^\alpha = I_i^\alpha(k_{\rho mn}, z, z')$ is the current at z due to a unit parallel current source at z',

$I_v^\alpha = I_v(k_{\rho mn}, z, z')$ is the current at z due to a unit series voltage source at z'.

$g_{\beta\gamma}^\alpha = g_{\beta\gamma}^\alpha\left(k_{\rho mn}, \rho-\rho'\right)$ is a function of radial distance from the source, and $\beta, \gamma = v, i$.

All of the above quantities depend on the Floquet wave indices (m,n), although this notation is suppressed in the summand of Eq. (3).

An important observation is that values for these transmission line quantities need only be calculated and stored for source and observation points at the *layer boundaries*. One can then calculate, in terms of these tabulated interface quantities, the *exact* voltage and current functions for each wavenumber $k_{\rho mn}$ and any z and z' using sinusoidal interpolation in z and z', obviating the need for tables for various values of z and z'. In this scheme, the voltage and reflection coefficient quantities, ($V_v^{TE}, V_i^{TE}, V_v^{TM}, V_i^{TM}, \overleftarrow{\Gamma}_i, \vec{\Gamma}_{i+1}$), must be stored for sampled values of k_ρ for each source/observation interface combination. It is not generally necessary to store the corresponding current quantities ($I_v^{TE}, I_i^{TE}, I_v^{TM}, I_i^{TM}$) since they can be obtained from the voltage values. Preliminary computations using this method have yielded a speed-up factor of about 5.

3. 3D interpolation for extracted terms (Ewald terms)

The extracted terms (the terms extracted from the total Green's function to regularize it) correspond to 2D periodic Green's functions in a homogeneous medium. These are calculated using the Ewald method. Although the Ewald method is a very efficient way to calculate the 2D periodic homogenous-medium Green's function, the calculation of this Green's function may also be speeded up by using interpolation. The Ewald spectral series is regular at the periodic source points, while the Ewald spatial series has singularities at the source points. If the spatial Ewald series is regularized, then both it and the spectral Ewald series can be easily interpolated. We regularize the spatial series by removing the contributions from the four corner sources of the (0,0) unit cell that contains the observation point. Interpolation of the regularized Ewald spatial series, combined with the Ewald spectral series contribution, requires a set of 3D interpolation tables since we have functional dependencies on $x-x'$, $y-y'$, and $z-z'$. Using this method, a speed-up factor of about 10 has been observed for the calculation of the 2D periodic homogenous-medium Green's function, using a simple quadratic interpolation with $3^3 = 27$ interpolation points. (A further speed-up may be possible by interpolating on a 3D simplex with 10 interpolation points.)

3. Conclusions

Various ways to use interpolation have been explored for accelerating the computation of the 2D periodic Green's function in a layered medium. The first method involved interpolation of the Green's function in the spatial domain, after regularizing it by extracting image terms corresponding to 2D periodic Green's functions in a homogenous medium. This interpolation is very fast, but it requires a 4D interpolation, and it also requires multiple interpolation tables since interpolation tables are needed for each combination of source and observation point layers, in order to avoid interpolating across layer boundaries.

An alternative is to interpolate in the spectral domain. In this case the regularized 2D periodic Green's function is computed "on the fly" by performing the summation over the required sample points in the spectral domain, as it would be without using interpolation. However, the spectral transmission line voltage and current quantities that are needed in the calculation of the summand are interpolated. These quantities are functions of the spectral radial wavenumber k_ρ as well as z and z'. However, the variations in z and z' may be accounted for exactly by using sinusoidal interpolation of the voltage and currents that are calculated when the source and observation points are located at the layer interfaces. Hence, multiple 1D tables (one for each different combination of source and observation point interfaces) are required, which store the voltage and current functions, as a function of the radial wavenumber.

In either method, the extracted terms corresponding to 2D periodic homogenous-medium Green's functions may also be accelerated by using the Ewald method together with interpolation. In this case the Ewald spatial series is accelerated by regularizing it and then both it and the Ewald spectral series are interpolated. The Ewald spatial series is regularized by removing contributions from the four corner source points in the unit cell.

4. References

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