A Property of the $L(c, \rho, n)$ Numbers and Its Application to Waveguide Propagation

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1. Introduction

The introduction of $K$, $L$, $M$ classes of real numbers is one of the characteristics of the formulated through the confluent hypergeometric functions [1] modern theory of $TE_{0n}$ modes in the azimuthally magnetized circular and coaxial ferrite waveguides [2-4]. In this paper the sameness of two kinds of $L$ numbers whose generating function is complex, resp. real, is established. Their relation to the normal, resp. slow $TE_{0n}$ waves in a coaxial configuration of the family mentioned determines the main physical result of the study – the existence of a line in the phase diagram in case of negative magnetization of the structure, at which the areas of propagation of both types of modes come into contact.

2. Theorem for the Identity of $L(c, \rho, n)$ and $\hat{L}(\hat{c}, \hat{\rho}, \hat{n})$ Numbers

Theorem 1: The statement of the theorem is expressed by the following three Lemmas.

Lemma 1: If $\chi^{(c)}_{k, a} (\rho)$ is the $n$th positive purely imaginary zero of the function $F(a, c; x, \rho) = \Phi(a, c; x) \Psi(a, c; \rho) + \Phi(a, c; \rho) \Psi(a, c; x)$ in $x \ (n = 1, 2, 3, ...)$ in which $\Phi(a, c; x)$ and $\Psi(a, c; x)$ are the Kummer and Tricomi confluent hypergeometric functions with $a = c/2 - jk$ – complex, $c = 3$, $x = jz$ – positive purely imaginary, $z$ – real, positive, $k = j(a - c/2) - real$, $-\infty < k < +\infty$, $\rho$ – real, positive, $0 < \rho < 1$, $K_{-} (c, \rho, n, k_{*}) = \left| a_{n} \right| \chi^{(c)}_{k, a} (\rho)$, $M_{-} (c, \rho, n, k_{*}) = \left| a_{n} \right| \chi^{(c)}_{k, a} (\rho)$, then the infinite sequences of positive real numbers $\left\{ \chi^{(c)}_{\nu, k, a} (\rho) \right\}$, $\left\{ K_{-} (c, \rho, n, k_{*}) \right\}$ and $\left\{ M_{-} (c, \rho, n, k_{*}) \right\}$ are convergent for $k_{*} \rightarrow -\infty$ ($n$ is fixed). The limit of the first sequence is zero and the limit of the second and third ones is the same. It equals the finite positive real number $L$ where $L = L(c, \rho, n)$. It holds:

$$\lim_{k_{*} \rightarrow -\infty} K_{-} (c, \rho, n, k_{*}) = \lim_{k_{*} \rightarrow -\infty} M_{-} (c, \rho, n, k_{*}) = L(c, \rho, n).$$

Lemma 2: If $\chi^{(c)}_{\nu, k, a} (\rho)$ is the $\nu$th positive real zero of the function $F(\hat{a}, \hat{c}; \hat{x}, \hat{\rho}) = \Phi(\hat{a}, \hat{c}; \hat{x}) \Psi(\hat{a}, \hat{c}; \hat{\rho}) + \Phi(\hat{a}, \hat{c}; \hat{\rho}) \Psi(\hat{a}, \hat{c}; \hat{x})$ in $\hat{x}$ in which $\Phi(\hat{a}, \hat{c}; \hat{x})$ and $\Psi(\hat{a}, \hat{c}; \hat{x})$ are the Kummer and Tricomi confluent hypergeometric functions, provided $\hat{a}$, $\hat{c}$, $\hat{x}$ are real, $\hat{a} < 0$, $\hat{a} < -\hat{n}$, $\hat{n} = 1, 2, 3, ...$, $\hat{c} = 3$, $(\hat{n} = 1, 2, ..., \hat{\rho} = abs(\hat{a})$, $\hat{k} = \hat{a} - \hat{c}/2$ – real, negative, $-\infty < \hat{k} < -\hat{c}/2$, $\hat{a} = \hat{c}/2 + \hat{k}$, $\hat{x} > 0$, $\hat{\rho}$ – real, positive, $0 < \hat{\rho} < 1$, $\hat{K}_{-} (\hat{c}, \hat{\rho}, \hat{n}, \hat{k}_{*}) = \left| \hat{a}_{n} \right| \chi^{(c)}_{\nu, k, a} (\hat{\rho})$, $\hat{M}_{-} (\hat{c}, \hat{\rho}, \hat{n}, \hat{k}_{*}) = \left| \hat{a}_{n} \right| \chi^{(c)}_{\nu, k, a} (\hat{\rho})$, then the infinite sequences of positive real numbers $\left\{ \chi^{(c)}_{\nu, k, a} (\hat{\rho}) \right\}$, $\left\{ \hat{K}_{-} (\hat{c}, \hat{\rho}, \hat{n}, \hat{k}_{*}) \right\}$ and $\left\{ \hat{M}_{-} (\hat{c}, \hat{\rho}, \hat{n}, \hat{k}_{*}) \right\}$ are convergent for $\hat{k}_{*} \rightarrow -\infty$ ($\hat{n}$ is fixed). The limit of the first sequence is zero and the limit of the second and third ones is the same. It equals the finite positive real number $\hat{L}$ where $\hat{L} = \hat{L}(\hat{c}, \hat{\rho}, \hat{n})$. It holds:

$$\lim_{\hat{k}_{*} \rightarrow -\infty} \hat{K}_{-} (\hat{c}, \hat{\rho}, \hat{n}, \hat{k}_{*}) = \lim_{\hat{k}_{*} \rightarrow -\infty} \hat{M}_{-} (\hat{c}, \hat{\rho}, \hat{n}, \hat{k}_{*}) = \hat{L}(\hat{c}, \hat{\rho}, \hat{n}).$$

Lemma 3: Under the conditions of Lemma 1 and Lemma 2, provided $c = \hat{c}$, $\rho = \hat{\rho}$, $n = \hat{n}$ and $k_{*} = \hat{k}_{*}$ – large negative, it is true:

$$\chi^{(c)}_{k, a} (\rho) = \hat{\chi}^{(c)}_{k, a} (\hat{\rho}),$$

$$\chi^{(c)}_{\nu, k, a} (\rho) = \hat{\chi}^{(c)}_{\nu, k, a} (\hat{\rho}).$$
\[ L(c, \rho, n) = \tilde{L}(\hat{c}, \hat{\rho}, \hat{n}). \quad (4) \]

The subscripts "+", "−" indicate quantities, corresponding to positive or negative \( k \) \((\hat{k})\).

Lemma 1 and Lemma 2 are proved numerically, harnessing the definition of \( \Phi(a,c;x) \) in terms of an infinite power series and the logarithmic representation of \( \Psi(a,c;x) \) [1]. The coincidence of results obtained for the same values of parameters chosen substitute Lemma 3, resp. Theorem 1 (cf. Table 1). Selected values of quantities \( L(c, \rho, n) \) are listed in Table 2.

Table 1: Numbers \( \lambda_{k,a}^{(c)}(\rho), K_{c}(\rho,n,p,k), M_{c}(\rho,n,p,k), \lambda_{k,a}^{(c)}(\hat{\rho}), \hat{K}_{c}(\hat{\rho},\hat{n},\hat{k}), \) and \( M_{c}(\hat{\rho},\hat{n},\hat{k}) \) for \( c = \hat{c} = 3, \rho = \hat{\rho} = 0.1 \) and \( 0.2, n = \hat{n} = 1 \) and large negative \( k = -\hat{k} \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \lambda_{k,a}^{(c)}(\rho) )</th>
<th>( K_{c}(\rho,n,p,k) )</th>
<th>( M_{c}(\rho,n,p,k) )</th>
<th>( \lambda_{k,a}^{(c)}(\hat{\rho}) )</th>
<th>( \hat{K}_{c}(\hat{\rho},\hat{n},\hat{k}) )</th>
<th>( M_{c}(\hat{\rho},\hat{n},\hat{k}) )</th>
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<tr>
<td>-10</td>
<td>0.75829 20695</td>
<td>7.58292 06953</td>
<td>7.66775 40201</td>
<td>0.77196 98723</td>
<td>7.71969 87233</td>
<td>6.56174 39148</td>
</tr>
<tr>
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<td>0.38165 52313</td>
<td>7.63310 50622</td>
<td>7.65454 30652</td>
<td>0.38336 43821</td>
<td>7.66728 76412</td>
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<td>-40</td>
<td>0.19114 63518</td>
<td>7.64585 40730</td>
<td>7.65122 81755</td>
<td>0.19135 99883</td>
<td>7.65439 95315</td>
<td>7.36735 95490</td>
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<tr>
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<td>0.12747 04016</td>
<td>7.64822 40930</td>
<td>7.65061 37897</td>
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<td>7.65202 20702</td>
<td>7.46072 15184</td>
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<tr>
<td>-80</td>
<td>0.09561 31784</td>
<td>7.64905 42747</td>
<td>7.65039 87168</td>
<td>0.09563 98830</td>
<td>7.65119 06364</td>
<td>7.50773 08119</td>
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<td>0.07649 43865</td>
<td>7.64943 86487</td>
<td>7.65029 91622</td>
<td>0.07650 80592</td>
<td>7.65080 59201</td>
<td>7.53604 38313</td>
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\( \rho = 0.1 \)

<table>
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<tr>
<th>( k )</th>
<th>( \lambda_{k,a}^{(c)}(\rho) )</th>
<th>( K_{c}(\rho,n,p,k) )</th>
<th>( M_{c}(\rho,n,p,k) )</th>
<th>( \lambda_{k,a}^{(c)}(\hat{\rho}) )</th>
<th>( \hat{K}_{c}(\hat{\rho},\hat{n},\hat{k}) )</th>
<th>( M_{c}(\hat{\rho},\hat{n},\hat{k}) )</th>
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<td>-10</td>
<td>0.98333 59223</td>
<td>9.83335 98229</td>
<td>9.94336 97572</td>
<td>1.01016 10507</td>
<td>10.10161 05069</td>
<td>8.58636 89308</td>
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<td>0.49635 23452</td>
<td>9.93064 70832</td>
<td>9.95853 78617</td>
<td>0.49988 27424</td>
<td>9.99765 48485</td>
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</tr>
<tr>
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<td>0.24888 91822</td>
<td>9.95556 72874</td>
<td>9.96256 48364</td>
<td>0.24930 79593</td>
<td>9.97231 83721</td>
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</tr>
<tr>
<td>-60</td>
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<td>0.09962 58923</td>
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\( \rho = 0.2 \)

Table 2: \( L(c, \rho, n) \) numbers in case \( c = 3 \) and \( n = 1,2,3 \) as a function of \( \rho \)

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( L(c, \rho, n) )</th>
<th>( \rho )</th>
<th>( L(c, \rho, n) )</th>
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</thead>
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<tr>
<td></td>
<td>( n = 1 )</td>
<td>( n = 2 )</td>
<td>( n = 3 )</td>
</tr>
<tr>
<td>0.00</td>
<td>6.5937</td>
<td>17.7125</td>
<td>33.7551</td>
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<tr>
<td>0.02</td>
<td>6.6575</td>
<td>18.2714</td>
<td>35.7818</td>
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<td>0.04</td>
<td>6.8167</td>
<td>19.3770</td>
<td>39.0196</td>
</tr>
<tr>
<td>0.06</td>
<td>7.0430</td>
<td>20.7240</td>
<td>42.5625</td>
</tr>
<tr>
<td>0.08</td>
<td>7.3229</td>
<td>22.2332</td>
<td>46.3417</td>
</tr>
<tr>
<td>0.10</td>
<td>7.6501</td>
<td>23.8818</td>
<td>50.3598</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \rho )</th>
<th>( L(c, \rho, n) )</th>
<th>( \rho )</th>
<th>( L(c, \rho, n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( n = 1 )</td>
<td>( n = 2 )</td>
<td>( n = 3 )</td>
<td>( n = 2 )</td>
</tr>
<tr>
<td>0.10</td>
<td>9.9639</td>
<td>34.3351</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Application to Waveguide Propagation

The zeros of function \( F(a,c;x,\rho) \) from Lemma 1 determine the eigenvalue spectrum \( \bar{B}_2 = \lambda_{k,a}^{(c)}(\rho)/(2\pi) \) of the coaxial ferrite waveguide with azimuthal magnetization for normal \( TE_{0n} \) modes, if it holds: \( k = a\beta/(2\bar{B}_2), \bar{B}_2 = (1 - \alpha^2 - \bar{B}_2^2)^{1/2}, x = x_0, x_0 = \mu_0 \omega e_r, x_0 = \mu_0 \omega e_r, \rho = \mu_0 \omega e_r, \beta = \beta/(\mu_0 \omega e_r), \bar{B}_2 = \beta/(\mu_0 \omega e_r), \bar{R}_0 = \beta/(\mu_0 \omega e_r), \bar{R}_0 = \beta/(\mu_0 \omega e_r), \alpha = \gamma M_0 / \omega \) is the off-diagonal element of Polder permeability tensor of the ferrite (\( \gamma \) = gyromagnetic ratio, \( M_0 \) = remanent magnetization, \( \omega \) = angular frequency of the wave), \( \beta \) is the phase constant of the wave, \( \beta_2 \) is the radial wave number, \( \beta_0 = \omega \sqrt{e_0 e_0} \) is the free space phase constant, \( e_r \) is the ferrite relative permittivity, \( r_0 \) and \( r_1 \) are the outer and inner conductor radii [4]. Similarly, it can be shown that the zeros of \( F(\hat{a},\hat{c};\hat{x},\hat{\rho}) \) from Lemma 2 specify the eigenvalue
The analysis reveals that the structure examined might guide three different rotationally symmetric TE modes with the subscript \( n = \hat{n} = 1 \): one normal and two slow, denoted by the symbols \( TE_{01} \), \( TE_{11}^{(1)} \) and \( TE_{11}^{(2)} \), resp. The relevant \( \beta_\perp, \beta_{\perp}^{(1)}(\tau_{01}) \) and \( \beta_{\perp}^{(2)}(\tau_{02}) \) phase characteristics are figured, applying the procedure, developed for normal \( TE_{0n} \) set of fields in a circular ferrite waveguide (\( \rho = 0 \)) \[2\] with \( \chi^{(c)}_{\perp, n}(\rho) \) for the first and with \( \beta_{k, \hat{n}}^{(c)}(\rho) \) for the second and third waves and are plotted in Figs. 1, 2, 3, resp., assuming \( \rho = \hat{\rho} = 0.2 \).

The solid lines in Fig. 1, labelled by \( \alpha_\perp > 0 \) correspond to positive and the dashed ones, marked by \( \alpha_\perp < 0 \) in Fig. 1 and by \( \alpha^{(1)} < 0 \) (\( |\alpha^{(1)}| < 1 \)) and by \( \alpha^{(2)} < 0 \) (\( |\alpha^{(2)}| > 1 \)) in Figs. 2, 3, resp. – to negative magnetization. (The unbroken curve in Fig. 1, denoted by \( \alpha = 0 \) is pertinent to the dielectric case.) The dashed lines in Fig. 1 are restricted from the side of higher frequencies by the \( E_{n-} \) – envelope with equation \( \beta_{en-} = \beta_{en-}(\tau_{en-}) \), written in parametric form as: \( \tau_{en-} = L(3, \rho, 1)/\left[\alpha_{en-} \left(1 - \alpha_{en-}^2\right)^{1/2}\right] \), \( \beta_{en-} = \left[1 - \alpha_{en-}^2\right]^{1/2} \), (\( \alpha_{en-} \) is the parameter). The existence of this curve is a direct corollary of Lemma 1. In the same way the characteristics in Fig. 2 are bounded from the side of lower frequencies by the \( E_{n-} \) – envelope whose equation is \( \beta_{en-}^{(1)} = \beta_{en-}(\tau_{en-}^{(1)}) \), where \( \tau_{en-}^{(1)} = \hat{L}(3, \hat{\rho}, 1)/\left[\alpha_{en-}^{(1)} \left(1 - \alpha_{en-}^{(1)}\right)^{2} \right]^{1/2} \), \( \beta_{en-}^{(1)} = \left[1 - \alpha_{en-}^{(1)}\right]^{3/2} \), (\( \alpha_{en-}^{(1)} \) is the parameter).

The appearance of this curve follows from Lemma 2. According to Lemma 3, if \( \rho = \hat{\rho} \) and \( \alpha_{en-} = \alpha_{en-}^{(1)} \), both envelopes coincide. Similar characteristic exists also in case \( \rho = 0 \) \[2\] (shown with dotted line in Fig. 2 only, to avoid overcrowding in Fig. 1). The abscissas of points (1), (2) (the minima of \( E_{n-} \) (\( \hat{E}_{n-} \) ) – curves ) are \( \min \tau_{en-} = 2L(3, \rho, 1) \) and \( \min \tau_{en-} = 2L(3, 1) \), provided \( \rho 
eq 0 \) and \( \rho = 0 \), resp., attained for \( \alpha_{en-} = \frac{1}{\sqrt{2}} \) at \( \beta_{en-} = \left[\alpha_{en-} \right] \) (\( L(3, 1) = 6.59365 41068 \) and \( L(3, \rho, 1) = 9.9639 \) for \( \rho = 0.2 \)), (see Figs. 1, 2 and Ref. [2]). (The superscripts (1), (2) fix the slow wave to which certain quantity relates, the subscripts “+”,”−” mark quantities, connected with positive or negative magnetization, resp., and the one “en−” is used to distinguish the parameters of the envelopes.) Summarizing, the existence of \( \hat{L}(\hat{c}, \hat{\rho}, n) \) and \( \hat{L}(\hat{e}, \hat{\rho}, \hat{n}) \) numbers determines appearance of the \( E_{n-} \) and \( \hat{E}_{n-} \) – characteristics and their identity for the same parameters – the coincidence of the latter. Physically this means that the \( E_{n-} \) (\( \hat{E}_{n-} \) ) – curve serves as a
dividing line which in case of negative magnetization separates the areas of propagation of normal $TE_{01}$ and slow $TE_{01}^{(1)}$ mode. The first lies to the left and the second to the right of it.

![Graph of curves](image)

Figure 2: $\tilde{\beta}^{(1)}(\tilde{\alpha}^{(1)})$ curves of slow $TE_{01}^{(1)}$ mode in the coaxial ferrite waveguide for $\tilde{\rho} = 0.2$ and $-1 < \tilde{\alpha}^{(1)} < 0$.

Figure 3: $\tilde{\beta}^{(2)}(\tilde{\alpha}^{(2)})$ curves of slow $TE_{01}^{(2)}$ mode in the coaxial ferrite waveguide for $\tilde{\rho} = 0.2$ and $-\infty < \tilde{\alpha}^{(2)} < -4.5674$.

4. Conclusion

The identity of two classes of the real numbers, connected with the complex, resp. real zeros of functions, composed by complex, resp. real confluent hypergeometric ones is substantiated. It is shown that this is equivalent to the existence of a characteristic, demarcating the regions in which normal and slow $TE_{0n}$ modes may get excited in a coaxial ferrite waveguide, if it is magnetized azimuthally in negative direction.

5. References


