A Property of the $L(c,\rho,n)$ Numbers and Its Application to Waveguide Propagation

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1. Introduction

The introduction of K, L, M classes of real numbers is one of the characteristics of the formulated through the confluent hypergeometric functions [1] modern theory of TE_{0n} modes in the azimuthally magnetized circular and coaxial ferrite waveguides [2-4]. In this paper the sameness of two kinds of L numbers whose generating function is complex, resp. real, is established. Their relation to the normal, resp. slow TE_{0n} waves in a coaxial configuration of the family mentioned determines the main physical result of the study – the existence of a line in the phase diagram in case of negative magnetization of the structure, at which the areas of propagation of both types of modes come into contact.

2. Theorem for the Identity of $L(c,\rho,n)$ and $\hat{L}(\hat{c},\hat{\rho},\hat{n})$ Numbers

Theorem 1: The statement of the theorem is expressed by the following three Lemmas.

Lemma 1: *If $\chi_{k,n}^{(c)}(\rho)$ is the nth positive purely imaginary zero of the function $F(a,c;x,\rho)$ = $\Phi(a,c;x)\Psi(a,c;\rho x)-\Phi(a,c;\rho x)\Psi(a,c;x)$ in x ($n=1,2,3...$) in which $\Phi(a,c;x)$ and $\Psi(a,c;x)$ are the Kummer and Tricomi confluent hypergeometric functions with $a=c/2-jk$ – complex, $c=3$, $x=jz$ – positive purely imaginary, z – real, positive, $k=j(a-c/2)$ – real, $-\infty<k<+\infty$, ρ – real, positive, $0<\rho<1$, $K_{\pm}(c,\rho,n,k_{\pm})=|k_{\pm}|\chi_{k_{\pm},n}^{(c)}(\rho)$, $M_{\pm}(c,\rho,n,k_{\pm})=|a_{\pm}|\chi_{k_{\pm},n}^{(c)}(\rho)$, then the infinite sequences of positive real numbers $\{\chi_{k_{-},n}^{(c)}(\rho)\}$, $\{K_{-}(c,n,\rho,k_{-})\}$ and $\{M_{-}(c,n,\rho,k_{-})\}$ are convergent for $k_{-}\to-\infty$ (n is fixed). The limit of the first sequence is zero and the limit of the second and third ones is the same. It equals the finite positive real number L where $L=L(c,\rho,n)$. It holds:*

$$\lim_{k_{-}\to-\infty} K_{-}(c,\rho,n,k_{-})=\lim_{k_{-}\to-\infty} M_{-}(c,\rho,n,k_{-})=L(c,\rho,n). \tag{1}$$

Lemma 2: *If $\hat{\chi}_{\hat{k},\hat{n}}^{(\hat{c})}(\hat{\rho})$ is the $\hat{n}$th positive real zero of the function $F(\hat{a},\hat{c};\hat{x},\hat{\rho})$ = $\Phi(\hat{a},\hat{c};\hat{x})\Psi(\hat{a},\hat{c};\hat{\rho}\hat{x})-\Phi(\hat{a},\hat{c};\hat{\rho}\hat{x})\Psi(\hat{a},\hat{c};\hat{x})$ in $\hat{x}$ in which $\Phi(\hat{a},\hat{c};\hat{x})$ and $\Psi(\hat{a},\hat{c};\hat{x})$ are the Kummer and Tricomi confluent hypergeometric functions, provided $\hat{a}$, $\hat{c}$, $\hat{x}$ are real, $\hat{a}<0$, $\hat{a}\neq-\hat{m}$, $\hat{m}=1,2,3,...$, $\hat{c}=3$, ($\hat{n}=1,2,...,\hat{p}$, $\hat{p}=abs[\hat{a}]$), $\hat{k}=\hat{a}-\hat{c}/2$ – real, negative, $-\infty<\hat{k}<-\hat{c}/2$, $\hat{a}=\hat{c}/2+\hat{k}$, $\hat{x}>0$, $\hat{\rho}$ – real, positive, $0<\hat{\rho}<1$, $\hat{K}_{-}(\hat{c},\hat{\rho},\hat{n},\hat{k}_{-})=|\hat{k}_{-}|\hat{\chi}_{\hat{k}_{-},\hat{n}}^{(\hat{c})}(\hat{\rho})$, $\hat{M}_{-}(\hat{c},\hat{\rho},\hat{n},\hat{k}_{-})=|\hat{a}_{-}|\hat{\chi}_{\hat{k}_{-},\hat{n}}^{(\hat{c})}(\hat{\rho})$, then the infinite sequences of positive real numbers $\{\hat{\chi}_{\hat{k}_{-},\hat{n}}^{(\hat{c})}(\hat{\rho})\}$, $\{\hat{K}_{-}(\hat{c},\hat{n},\hat{\rho},\hat{k}_{-})\}$ and $\{\hat{M}_{-}(\hat{c},\hat{n},\hat{\rho},\hat{k}_{-})\}$ are convergent for $\hat{k}_{-}\to-\infty$ ($\hat{n}$ is fixed). The limit of the first sequence is zero and the limit of the second and third ones is the same. It equals the finite positive real number $\hat{L}$ where $\hat{L}=\hat{L}(\hat{c},\hat{\rho},\hat{n})$. It holds:*

$$\lim_{\hat{k}_{-}\to-\infty} \hat{K}_{-}(\hat{c},\hat{\rho},\hat{n},\hat{k}_{-})=\lim_{\hat{k}_{-}\to-\infty} \hat{M}_{-}(\hat{c},\hat{\rho},\hat{n},\hat{k}_{-})=\hat{L}(\hat{c},\hat{\rho},\hat{n}). \tag{2}$$

Lemma 3: *Under the conditions of Lemma 1 and Lemma 2, provided $c=\hat{c}$, $\rho=\hat{\rho}$, $n=\hat{n}$ and $k_{-}=\hat{k}_{-}$ – large negative, it is true:*

$$\chi_{k_{-},n}^{(c)}(\rho)\approx\hat{\chi}_{\hat{k}_{-},\hat{n}}^{(\hat{c})}(\hat{\rho}), \tag{3}$$

$$L(c,\rho,n) = \hat{L}(\hat{c},\hat{\rho},\hat{n}). \tag{4}$$

The subscripts "+", "–" indicate quantities, corresponding to positive or negative k ($\hat{k}$).

Lemma 1 and Lemma 2 are proved numerically, harnessing the definition of $\Phi(a,c;x)$ in terms of an infinite power series and the logarithmic representation of $\Psi(a,c;x)$ [1]. The coincidence of results obtained for the same values of parameters chosen substantiates Lemma 3, resp. Theorem 1 (cf. Table 1). Selected values of quantities $L(c,\rho,n)$ are listed in Table 2.

Table 1: Numbers $\chi_{k,n}^{(c)}(\rho)$, $K_-(c,n,\rho,k_-)$, $M_-(c,n,\rho,k_-)$, $\hat{\chi}_{\hat{k}_-,\hat{n}}^{(\hat{c})}(\hat{\rho})$, $\hat{K}_-(\hat{c},\hat{\rho},\hat{n},\hat{k}_-)$ and $\hat{M}_-(\hat{c},\hat{\rho},\hat{n},\hat{k}_-)$ for $c=\hat{c}=3$, $\rho=\hat{\rho}=0.1$ and 0.2, $n=\hat{n}=1$ and large negative $k_-=\hat{k}_-$

k_-	$\chi_{k_-,n}^{(c)}(\rho)$	$K_-(c,\rho,n,k_-)$	$M_-(c,\rho,n,k_-)$	$\chi_{\hat{k}_-,\hat{n}}^{(\hat{c})}(\hat{\rho})$	$\hat{K}_-(\hat{c},\hat{\rho},\hat{n},\hat{k}_-)$	$\hat{M}_-(\hat{c},\hat{\rho},\hat{n},\hat{k}_-)$
	$\rho=0.1$					
-10	0.75829 20695	**7.**58292 06953	**7.6**6775 40201	0.77196 98723	**7.**71969 87233	6.56174 39148
-20	0.38165 52531	**7.6**3310 50622	**7.65**454 30652	0.38336 43821	**7.6**6728 76412	**7.**09224 10681
-40	0.19114 63518	**7.6**4585 40730	**7.65**122 81755	0.19135 99883	**7.65**439 95315	**7.**36735 95490
-60	0.12747 04016	**7.6**4822 40930	**7.650**61 37897	0.12753 37012	**7.65**202 20702	**7.**46072 15184
-80	0.09561 31784	**7.6**4905 42747	**7.650**39 87168	0.09563 98830	**7.65**119 06364	**7.**50773 08119
-100	0.07649 43865	**7.6**4943 86487	**7.650**29 91622	0.07650 80592	**7.650**80 59201	**7.**53604 38313
	$\rho=0.2$					
-10	0.98333 59823	**9.**83335 98229	**9.9**4336 97572	1.01016 10507	10.10161 05069	8.58636 89308
-20	0.49653 23542	**9.9**3064 70832	**9.9**5853 78617	0.49988 27424	**9.9**9765 48485	**9.**24783 07348
-40	0.24888 91822	**9.9**5556 72874	**9.96**256 48364	0.24930 79593	**9.9**7231 83721	**9.**59835 64331
-60	0.16600 34874	**9.96**020 92426	**9.96**332 13218	0.16612 75691	**9.96**765 41488	**9.**71846 27951
-80	0.12452 29493	**9.96**183 59433	**9.96**358 68934	0.12457 52963	**9.96**602 37011	**9.**77916 07567
-100	0.09962 58923	**9.96**258 92279	**9.96**370 99562	0.09965 26939	**9.96**526 93925	**9.**81579 03517

Table 2: $L(c,\rho,n)$ numbers in case $c=3$ and $n=1,2,3$ as a function of ρ

ρ	$L(c,\rho,n)$			ρ	$L(c,\rho,n)$		
	$n=1$	$n=2$	$n=3$		$n=1$	$n=2$	$n=3$
0.00	6.5937	17.7125	33.7551	0.10	7.6501	23.8818	50.3598
0.02	6.6575	18.2714	35.7818	0.12	8.0218	25.6673	54.6213
0.04	6.8167	19.3770	39.0196	0.14	8.4374	27.5943	59.2064
0.06	7.0430	20.7240	42.5625	0.16	8.8981	29.6709	64.1167
0.08	7.3229	22.2332	46.3417	0.18	9.4058	31.9126	69.5189
0.10	7.6501	23.8818	50.3598	0.20	9.9639	34.3351	75.3475

3. Application to Waveguide Propagation

The zeros of function $F(a,c;x,\rho)$ from Lemma 1 determine the eigenvalue spectrum $\bar{\beta}_2 = \chi_{k,n}^{(c)}(\rho)/(2\bar{r}_0)$ of the coaxial ferrite waveguide with azimuthal magnetization for normal TE_{0n} modes, if it holds: $k = \alpha\bar{\beta}/(2\bar{\beta}_2)$, $\bar{\beta}_2 = \left(1-\alpha^2-\bar{\beta}^2\right)^{1/2}$, $x=x_0$, $x_0=jz_0$, $z_0=2\bar{\beta}_2\bar{r}_0$, $\rho=\bar{r}_1/\bar{r}_0$, $\bar{\beta}=\beta/\left(\beta_0\sqrt{\varepsilon_r}\right)$, $\bar{\beta}_2=\beta_2/\left(\beta_0\sqrt{\varepsilon_r}\right)$, $\bar{r}_0=\beta_0 r_0\sqrt{\varepsilon_r}$, $\bar{r}_1=\beta_0 r_1\sqrt{\varepsilon_r}$, $\alpha=\gamma M_r/\omega$ is the off-diagonal element of Polder permeability tensor of the ferrite (γ – gyromagnetic ratio, M_r – remanent magnetization, ω – angular frequency of the wave), β is the phase constant of the wave, β_2 is the radial wave number, $\beta_0=\omega\sqrt{\varepsilon_0\mu_0}$ is the free space phase constant, ε_r is the ferrite relative permittivity, r_0 and r_1 are the outer and inner conductor radii [4]. Similarly, it can be shown that the zeros of $F(\hat{a},\hat{c};\hat{x},\hat{\rho})$ from Lemma 2 specify the eigenvalue

spectrum $\bar{\hat{\beta}}_2 = \hat{\chi}^{(\hat{c})}_{\hat{k},\hat{n}}(\hat{\rho})/(2\bar{\hat{r}}_0)$ of the configuration considered for slow $T\hat{E}_{0\hat{n}}$ modes, provided $\hat{k} = \hat{\alpha}\bar{\hat{\beta}}/(2\bar{\hat{\beta}}_2)$, $\bar{\hat{\beta}}_2 = \left[\bar{\hat{\beta}}^2 - (1-\hat{\alpha}^2)\right]^{1/2}$, $\hat{x} = \hat{x}_0$, $\hat{x}_0 = 2\bar{\hat{\beta}}_2\bar{\hat{r}}_0$, $\hat{\rho} = \bar{\hat{r}}_1/\bar{\hat{r}}_0$, $\bar{\hat{\beta}} = \hat{\beta}/(\beta_0\sqrt{\varepsilon_r})$, $\bar{\hat{\beta}}_2 = \hat{\beta}_2/(\beta_0\sqrt{\varepsilon_r})$, $\bar{\hat{r}}_0 = \beta_0\hat{r}_0\sqrt{\varepsilon_r}$, $\bar{\hat{r}}_1 = \beta_0\hat{r}_1\sqrt{\varepsilon_r}$. The sense of quantities $\hat{r}_0$, $\hat{r}_1$, $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\beta}_2$ is the same as that of the ones without hats "^".

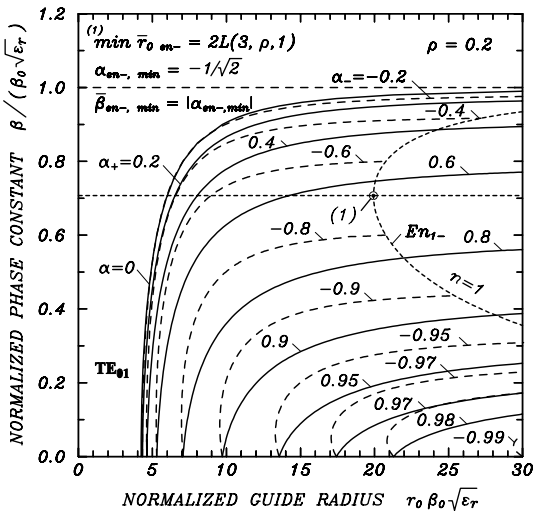

Figure 1: $\beta_\pm(\bar{r}_{0\pm})$ curves of the normal TE_{01} mode in the coaxial ferrite waveguide for $\rho = 0.2$.

The analysis reveals that the structure examined might guide three different rotationally symmetric TE modes with the subscript $n = \hat{n} = 1$: one normal and two slow, denoted by the symbols TE_{01}, $T\hat{E}^{(1)}_{01}$ and $T\hat{E}^{(2)}_{01}$, resp. The relevant $\bar{\beta}_\pm(\bar{r}_{0\pm})$, $\bar{\hat{\beta}}^{(1)}_-(\bar{\hat{r}}^{(1)}_{0-})$ and $\bar{\hat{\beta}}^{(2)}_-(\bar{\hat{r}}^{(2)}_{0-})$ phase characteristics are figured, applying the procedure, developed for normal TE_{0n} set of fields in a circular ferrite waveguide ($\rho = 0$) [2] with $\chi^{(c)}_{k_\pm,n}(\rho)$ for the first and with $\hat{\chi}^{(\hat{c})}_{\hat{k}_-,\hat{n}}(\hat{\rho})$ for the second and third waves and are plotted in Figs. 1, 2, 3, resp., assuming $\rho = \hat{\rho} = 0.2$. The solid lines in Fig. 1, labelled by $\alpha_+ > 0$ correspond to positive and the dashed ones, marked by $\alpha_- < 0$ in Fig. 1 and by $\hat{\alpha}^{(1)}_- < 0$ ($\left|\hat{\alpha}^{(1)}_-\right| < 1$) and by $\hat{\alpha}^{(2)}_- < 0$ ($\left|\hat{\alpha}^{(2)}_-\right| > 1$) in Figs. 2, 3, resp. − to negative magnetization. (The unbroken curve in Fig. 1, denoted by $\alpha = 0$ is pertinent to the dielectric case.) The dashed lines in Fig. 1 are restricted from the side of higher frequencies by the En_{1-} − envelope with equation $\bar{\beta}_{en-} = \bar{\beta}_{en-}(\bar{r}_{0en-})$, written in parametric form as: $\bar{r}_{0en-} = L(3,\rho,1)/\left[|\alpha_{en-}|\left(1-\alpha^2_{en-}\right)^{1/2}\right]$, $\bar{\beta}_{en-} = \left(1-\alpha^2_{en-}\right)^{1/2}$, (α_{en-} is the parameter). The existence of this curve is a direct corollary of Lemma 1. In the same way the characteristics in Fig. 2 are bounded from the side of lower frequencies by the $\hat{E}n_{1-}$ − envelope whose equation is $\bar{\hat{\beta}}^{(1)}_{en-} = \bar{\hat{\beta}}^{(1)}_{en-}(\bar{\hat{r}}^{(1)}_{0en-})$, where $\bar{\hat{r}}^{(1)}_{0en-} = \hat{L}(3,\hat{\rho},1)/\left\{\left|\hat{\alpha}^{(1)}_{en-}\right|\left[1-\left(\hat{\alpha}^{(1)}_{en-}\right)^2\right]^{1/2}\right\}$, $\bar{\hat{\beta}}^{(1)}_{en-} = \left[1-\left(\hat{\alpha}^{(1)}_{en-}\right)^2\right]^{1/2}$, ($\hat{\alpha}^{(1)}_{en-}$ is the parameter). The appearance of this curve follows from Lemma 2. According to Lemma 3, if $\rho \equiv \hat{\rho}$ and $\alpha_{en-} \equiv \hat{\alpha}^{(1)}_{en-}$, both envelopes coincide. Similar characteristic exists also in case $\rho = 0$ [2] (shown with dotted line in Fig. 2 only, to avoid overcrowding in Fig. 1). The abscissas of points (1), (2) (the minima of En_{1-} ($\hat{E}n_{1-}$) – curves) are $min\,\bar{r}_{0en-} = 2L(3,\rho,1)$ and $min\,\bar{r}_{0en-} = 2L(3,1)$, provided $\rho \neq 0$ and $\rho = 0$, resp., attained for $\alpha_{en-,min} = -1/\sqrt{2}$ at $\bar{\beta}_{en-,min} = |\alpha_{en-,min}|$ ($L(3,1) =$ 6.59365 41068 and $L(3,\rho,1) =$ 9.9639 for $\rho = 0.2$), (see Figs. 1,2 and Ref. [2]). (The superscripts (1), (2) fix the slow wave to which certain quantity relates, the subscripts "+","–" mark quantities, connected with positive or negative magnetization, resp., and the one "$en-$" is used to distinguish the parameters of the envelopes.) Summarizing, the existence of $L(c,\rho,n)$ and $\hat{L}(\hat{c},\hat{\rho},\hat{n})$ numbers determines appearance of the En_{1-} and $\hat{E}n_{1-}$ – characteristics and their identity for the same parameters – the coincidence of the latter. Physically this means that the En_{1-} ($\hat{E}n_{1-}$) – curve serves as a

dividing line which in case of negative magnetization separates the areas of propagation of normal TE_{01} and slow $T\hat{E}_{01}^{(1)}$ mode. The first lies to the left and the second to the right of it.

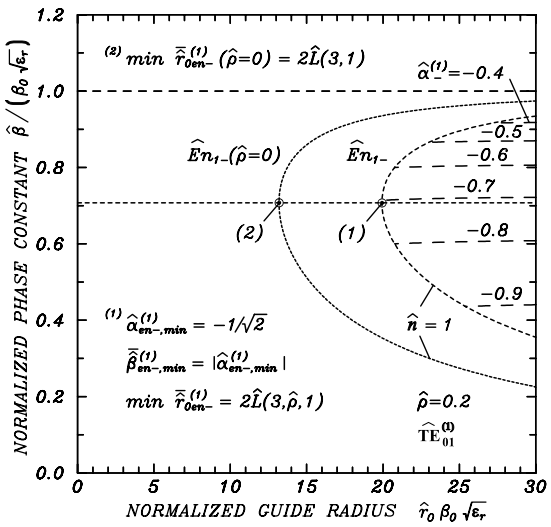

Figure 2: $\bar{\hat{\beta}}_{-}^{(1)}\left(\bar{\hat{r}}_{0-}^{(1)}\right)$ curves of slow $T\hat{E}_{01}^{(1)}$ mode in the coaxial ferrite waveguide for $\hat{\rho} = 0.2$ and $-1 < \hat{\alpha}_{-}^{(1)} < 0$.

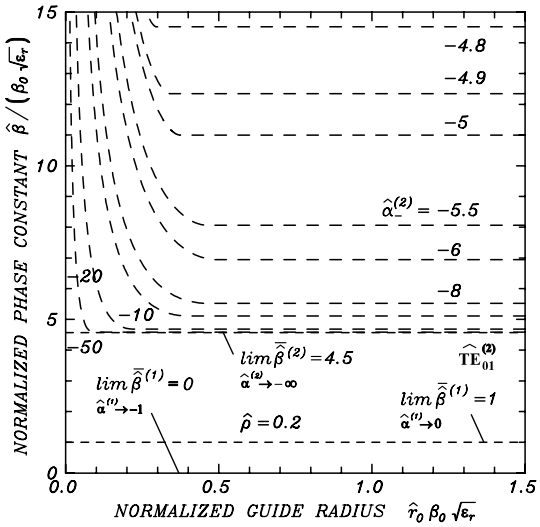

Figure 3: $\bar{\hat{\beta}}_{-}^{(2)}\left(\bar{\hat{r}}_{0-}^{(2)}\right)$ curves of slow $T\hat{E}_{01}^{(2)}$ mode in the coaxial ferrite waveguide for $\hat{\rho} = 0.2$ and $-\infty < \hat{\alpha}_{-}^{(2)} < -4.5674$.

4. Conclusion

The identity of two classes of the real numbers, connected with the complex, resp. real zeros of functions, composed by complex, resp. real confluent hypergeometric ones is substantiated. It is shown that this is equivalent to the existence of a characteristic, demarcating the regions in which normal and slow TE_{0n} modes may get excited in a coaxial ferrite waveguide, if it is magnetized azimuthally in negative direction.

5. References

1. F.G. Tricomi, *Fonctions Hypergéométrique Confluentes*, Paris, France: Gauthier-Villars, 1960.

2. G. N. Georgiev and M. N. Georgieva-Grosse, "A new property of the complex Kummer function and its application to waveguide propagation," *IEEE Antennas Wireless Propagation Lett.*, vol. AWPL-2, Dec. 2003, pp. 306-309.

3. G. N. Georgiev and M. N. Georgieva-Grosse, "An approximate method for analysis of azimuthally magnetized circular ferrite waveguide phase shifter", in *Proc. 2006 1st Europ. Conf. Antennas Propagation EUCAP 2006,* Nice, France, Nov. 6-10, 2006, in CD.

4. G. N. Georgiev and M. N. Georgieva-Grosse, "An iterative method for analysis of coaxial ferrite waveguide phase shifter with azimuthal magnetation", in *Proc. 29th ESA Antenna Worksh. Multiple Beams Reconfigurable Antennas,* ESA/ESTEC, Noordwijk, The Netherlands, Apr. 18-20, 2007, pp. 252-255, also in CD.