A Multilevel Direct Solver for Quasi-Planar Scatterers Based on the Non-Uniform Grid Approach

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Abstract

We introduce a multilevel direct solver for scattering from quasi-planar objects. The solver relies on the compression of the interactions between distinct domains. The compression is performed in three steps: field compression using the non-uniform grid approach, current compression using the rank revealing QR decomposition separating local from interacting currents, and local problem solution based on Schur's complement. The interacting currents are repeatedly aggregated with the neighboring sections in a multilevel process. The resulting compressed system of equations is solved directly. The algorithm attains $O(N^{1.5}\log^3 N)$ complexity with N being the number of unknowns.

1. Introduction

Algorithms for the numerical analysis of electromagnetic scattering from planar or quasi-planar objects are of great practical interest. Potential applications of such algorithms include the analysis of printed circuit boards, integrated circuit interconnects, as well as large phased-array antennas. Scattering from a quasi-planar object is often formulated in terms of the electric field integral equation (EFIE), which can be subsequently converted to a system of linear equations using the method of moments (MoM). The straightforward solution of the MoM linear system of equations becomes impractical as the electrical size of the object increases, because of the $O(N^3)$ computational complexity, where N denotes the number of unknowns. A number of fast iterative algorithms that drastically accelerate the iterative solution of the MoM matrix equations have been proposed. Unfortunately, with few exceptions, iterative solvers only consider one 'right-hand-side' at a time, which makes them less attractive when multiple excitations have to be considered. Also, iterative solvers are adversely affected by poorly conditioned equations of resonant problems. On the other hand, direct solvers for electromagnetic problems facilitate repeated solutions for various excitations/incident fields. In the past, efficient direct solvers for elongated geometries have been proposed in [1]. Also fast direct solvers for quasi-static and moderately high frequency regimes have been developed in [2-3]. Following the ideas of earlier works, we present a multilevel algorithm which relieves the computational burden associated with the solution of quasi-planar scattering problems.

2. Formulation

Consider a quasi-planar scatterer, whose height is significantly smaller than its lateral dimensions and the wavelength of the incident field. For simplicity, we consider an MoM formulation employing the EFIE and Galerkin choice of basis and testing functions, thus, obtaining a symmetric system of linear equations: $\mathbf{Z}\,\mathbf{J}=\mathbf{E}$. Here $\mathbf{J}$ and $\mathbf{E}$ denote the N-vectors of unknown current expansion coefficients and the incident field samples, respectively. and $\mathbf{Z}$ is the $N\times N$ generalized impedance matrix.

As a preliminary step, the scatterer is partitioned into a multilevel hierarchy of domains described by a quad tree. The finest level domains comprise at most a preset number of unknowns. For each finest level domain, we seek a reduced current expansion basis that is sufficient to reproduce the field radiated by this domain onto the outer domain. Currents $\mathbf{J}_A$ in A produce outside this domain a partial field, which is expressed as, $\mathbf{E}_B=\mathbf{Z}_A^B\mathbf{J}_A$,where $\mathbf{Z}_A^B$ is the corresponding off-diagonal block of the MoM matrix. The far zone (FZ) is defined as all observation points outside an FZ-circle of A of radius ΩR, where R is the minimal radius of a circumscribing circle of A and $\Omega > 1$ is a parameter. The near zone (NZ) is the complementary area of FZ on the scatterer outside A. The field in FZ and NZ shall be sampled on an analytic non-uniform grid (NG) and precomputed grid (PG), respectively, as described below.

The FZ field $\mathbf{E}_B$, can be described with an arbitrary precision if interpolated from some efficient sampling grid. In our algorithm, the field in the FZ is sampled at the NG points [4]. The NG sampling strategy is based on the observation that the field radiated by a finite size source behaves, locally, as a bandlimited function of the angle and the radial distance multiplied by a common phase factor. This behavior is exploited in devising an efficient sampling scheme for the far field of subdomain A. Indeed the radiated field is sampled on an NG and the radiated field on the surface is interpolated from the samples on the NG. According to the equivalence principle, the NG does not necessary need to be on the scatterer itself, but may be located on some enclosing surface. The number of NG samples needed to describe the FZ fields for the quasi-planar scatterer is proportional to the perimeter of the subdomain multiplied by some logarithmic function of the problem size.

The authors are not aware of any compact sampling scheme for the NZ on a quasi-planar surface, however, a numerical examination shows that the number of degrees of freedom needed to describe the NZ field in a quasi-planar regime is actually proportional to the perimeter of A and some logarithmically weak function. Therefore, we propose a numerical method for determining the optimal sampling grid in the NZ using the rank revealing QR decomposition (RRQR).

The single block compression consists of three steps: Field samples compression using the preceding definitions, compression of current sources residing in subdomain A using RRQR and compression of the local interaction of currents and field inside A using Schur's complement method, see Figure 1.

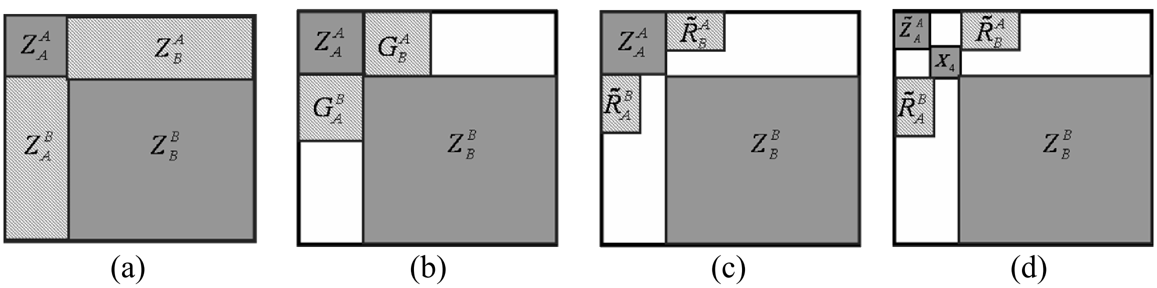

Figure 1. a) Separation into two domains. b) Off-diagonal block compression by grid sampling. c) Compression of the current sources using RRQR. d) Interacting and local part separation in sub-domain, using Schur's complement.

For the first step, we may use the preceding definitions to describe the off-diagonal interaction matrix as, $\mathbf{Z}_A^B = \mathbf{T}^B \mathbf{G}_A^B + \mathcal{E}$ where $\mathcal{E}$ represents the united interpolation and will be neglected. For the second step, matrix $\mathbf{G}_A^B$ is decomposed using the RRQR decomposition on the column space, $(\mathbf{G}_A^B)^\dagger \mathbf{P} = \mathbf{QR}$ where the dagger denotes the Hermitian conjugate. The RRQR reveals the numerical rank of $\mathbf{G}_A^B$ (and consequently of $\mathbf{Z}_A^B$) which can be interpreted as the least number of basis functions of $\mathbf{Q}$ that contribute to the field in B. The rest of these functions do not interact with the outer domain. This classification is used to provide the compression of the current sources by right-multiplication of block $\mathbf{Z}_A^B$ by $\mathbf{Q}$. We also perform a basis transformation on the current vector, $\mathbf{J}_A = \mathbf{Q}\bar{\mathbf{J}}_A$, such that $\mathbf{E}_B = \mathbf{T}^B \mathbf{P} \mathbf{R}^\dagger \bar{\mathbf{J}}_A$. Let's denote the interacting and local basis $\mathbf{Q} = [\tilde{\mathbf{Q}} \vdots \hat{\mathbf{Q}}]$ and $\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{R}_{22} \end{bmatrix}$, where, again, the tilde sign denotes interacting functions and the arc denotes the non-interacting ones. The $\mathbf{R}_{22}$ block is rank deficient and hence shall be truncated as the foundation for this compression step, $\mathbf{R} \approx \left[\tilde{\mathbf{R}}^T \vdots\ 0 \right]^T$, where $\tilde{\mathbf{R}} = \left[\mathbf{R}_{11} \vdots \mathbf{R}_{12} \right]$. The corresponding current vector is $\bar{\mathbf{J}}_A = [\tilde{\mathbf{J}}_A^T \vdots \hat{\mathbf{J}}_A^T]^T$. Thanks to the symmetric EFIE formulation, the same compression is applicable to $\mathbf{Z}_B^A = (\mathbf{Z}_A^B)^T$. Hence, the testing functions corresponding to the non-interacting basis are also unaffected by any field radiated by the currents from the outer domain. The field inside the source domain can be expressed as $\mathbf{E}_A = \mathbf{Z}_A^A \mathbf{J}_A + \mathbf{Z}_B^A \mathbf{J}_B$ where $\mathbf{Z}_A^A$ is the diagonal block of the MoM matrix corresponding to subdomain A and $\mathbf{J}_B$ are the currents in the outer domain. Applying the transformation suggested earlier and left-multiplying by $\mathbf{Q}^T$, we perform a symmetric testing and basis function transformation which yields $\bar{\mathbf{E}}_A = \bar{\mathbf{Z}}_A^A \bar{\mathbf{J}}_A + \mathbf{Q}^T \mathbf{Z}_B^A \mathbf{J}_B$ where $\bar{\mathbf{Z}}_A^A = \mathbf{Q}^T \mathbf{Z}_A^A \mathbf{Q}$ and $\bar{\mathbf{E}}_A = \mathbf{Q}^T \mathbf{E}_A$ is the field, which is decomposed into global and local components: $\bar{\mathbf{E}}_A = [\breve{\mathbf{E}}_A^T \vdots \widehat{\mathbf{E}}_A^T]^T$. We also partition $\bar{\mathbf{Z}}_A^A$, $\bar{\mathbf{Z}}_A^A = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \\ \mathbf{X}_3 & \mathbf{X}_4 \end{bmatrix}$ and get two coupled sets equations

$$\begin{aligned} \breve{\mathbf{E}}_A &= \mathbf{X}_1 \tilde{\mathbf{J}}_A + \mathbf{X}_2 \widehat{\mathbf{J}}_A + \tilde{\mathbf{R}}^* \left(\mathbf{T}^B \mathbf{P}\right)^T \mathbf{J}_B \\ \widehat{\mathbf{E}}_A &= \mathbf{X}_3 \tilde{\mathbf{J}}_A + \mathbf{X}_4 \widehat{\mathbf{J}}_A \end{aligned} \tag{1}$$

The second equation set of (1) represents the local interactions which do not depend, directly, on the outer current $\mathbf{J}_B$. Thus, in the last step, we formally solve the local currents $\widehat{\mathbf{J}}_\mathbf{A} = \mathbf{X}_4^{-1} \widehat{\mathbf{E}}_\mathbf{A} - \mathbf{X}_4^{-1} \mathbf{X}_3 \tilde{\mathbf{J}}_\mathbf{A}$ and substitute it into the first equation set, yielding $\tilde{\mathbf{E}}_A = \tilde{\mathbf{Z}}_A^A \tilde{\mathbf{J}}_A + \tilde{\mathbf{R}}^* \left(\mathbf{T}^B \mathbf{P}\right)^T \mathbf{J}_B$. Here we define $\tilde{\mathbf{E}}_A = [\breve{\mathbf{E}}_A - \mathbf{X}_2 \mathbf{X}_4^{-1} \widehat{\mathbf{E}}_A]$ as the effective local incident field and $\tilde{\mathbf{Z}}_A^A = [\mathbf{X}_1 - \mathbf{X}_2 \mathbf{X}_4^{-1} \mathbf{X}_3]$, which is known as the Schur complement of $\mathbf{X}_4$. It is seen that once $\tilde{\mathbf{J}}_A$ is found, we can return to determine the local current coefficients, $\widehat{\mathbf{J}}_A$. The whole compression process for the interacting field can be described using the following transformation matrix, $\tilde{\mathbf{V}} = \tilde{\mathbf{Q}}^T - \mathbf{X}_2 \mathbf{X}_4^{-1} \widehat{\mathbf{Q}}^T$, such that $\tilde{\mathbf{E}}_A = \tilde{\mathbf{V}} \mathbf{E}_A$, whereas for compressing the radiating currents we used $\tilde{\mathbf{J}}_A = \tilde{\mathbf{Q}}_A \mathbf{J}_A$.

The procedure of single block compression is performed for every subdomain at the finest level so that the total number of interacting unknowns is reduced. Now we would like to take every four neighboring blocks and aggregate them together to build a new set of blocks for the lower level, $A \in \bigcup_{j=1}^{4} A_j$, here we abuse the notation of previously defined subdomain A, so now it represents the new aggregated domain composed of four children $A_{1..4}$. This new block inherits only the interacting currents from the previous level, and its interaction with the rest of the scatterer shall be examined using the same procedures as developed in the previous chapter. The new local interaction matrix, which corresponds to the interaction of the inherited radiating currents with the local fields in the new domain, is built. Only the interacting functions are taken into account in the build up of the new local matrix. A non-uniform grid is built and the interaction of the basis with the new grid is found by interpolating from every grid of the four neighbors to the new grid. Subsequently, RRQR is performed on the new interaction matrix and the currents are once again transformed into a new basis while separating the interacting and local functions. Next, the local interaction matrix is compressed using the Schur's complement representation. This process of domain aggregation continues recursively until there remains only one root domain. The multilevel impedance matrix compression forms the basis for the solution process to be described in the next section. The remaining currents of the root domain shall be solved for directly by using the LU decomposition.

The incident field over the scatterer, $\mathbf{E}$, is first decomposed for the finest level domains. Following the testing function transformations suggested earlier, the field is split into the interacting and local parts, denoted by the tilde and arc signs, respectively. In the multilevel process, the interacting fields are aggregated as a preamble to the subsequent transformations until they reach the root level, which is denoted by $\widehat{\mathbf{E}}_\mathbf{1}^\mathbf{0}$. At the root level, the currents are solved for directly using the LU decomposition of the impedance matrix, which is the result of the multi-level compression process described in the previous section. We have $\left[\widehat{\mathbf{Z}}^0\right] \widehat{\mathbf{J}}_1^0 = \widehat{\mathbf{E}}_1^0$ where the superscript denotes the level and the subscript denotes the domain in the particular level. The matrix $\widehat{\mathbf{Z}}^\mathbf{0}$ is the root level local interaction matrix. The root level currents $\widehat{\mathbf{J}}_\mathbf{1}^\mathbf{0}$ are decomposed into the interacting currents of the preceding level $\widehat{\mathbf{J}}_1^{0^T} = [\tilde{\mathbf{J}}_1^{1^T} \vdots \tilde{\mathbf{J}}_2^{1^T} \vdots \tilde{\mathbf{J}}_3^{1^T} \vdots \tilde{\mathbf{J}}_4^{1^T}]^T$. For each of these interacting currents, we may find the local counterpart, to complete the current set in each subdomain of the root level. Subsequently, each of these domains is decomposed into its preceding level by aggregating the local and radiating functions followed by inverse basis transformation. For example, for the first subdomain of level *m*, we have $(\mathbf{Q}_A^m)^\dagger [\tilde{\mathbf{J}}_1^{m^T} \vdots \widehat{\mathbf{J}}_1^{m^T}]^T = [\tilde{\mathbf{J}}_1^{m+1^T} \vdots \tilde{\mathbf{J}}_2^{m+1^T} \vdots \tilde{\mathbf{J}}_3^{m+1^T} \vdots \tilde{\mathbf{J}}_4^{m+1^T}]^T$. This process is continued until the finest level is reached and all the currents are computed.

Towards estimating the computational complexity, we suppose that the domains of a given level are all equal in size. Let level m comprise N/N_m domains each described by N_m basis functions. The number of NG points and radiating basis functions would be proportional to the perimeter of this domain, i.e., $N_{NG} = O(N_m^{1/2}\log N_m)$. The calculation of the RRQR would require $O(N_{NG}^3) = O(N_m^{3/2}\log^3 N_m)$ operations. Therefore each level's computational complexity is of $O(NN_m^{1/2}\log^3 N_m)$. Since we used a quad tree decomposition, $N_m = 4N_{m-1}$ and the number of levels is $M = \log_4(N/N_M)$, where N_M is the number of points on the finest level. Hence the compression algorithm computational complexity is $O(N^{3/2}\log^3 N)$. After the compression is completed, we are still left with radiating unknowns which remain to be solved for directly. We end up with the maximum of $O(2N_M^{1/2}\log N_M)$ radiating basis functions for each domain. Also, the compression at each level is at least by a factor of 2. Continuing this process, we obtain that at the coarsest level we end up with $O(2^M N_M^{1/2}\log N_M)$ unsolved unknowns. Substituting $N_M = N/4^M$, the number of unresolved expansion coefficients is of $O(N^{1/2}\log N)$. Any direct solution of this problem takes $O(N^{3/2}\log^3 N)$ computational operations, which is comparable with the compression scheme described before. Therefore the overall complexity of the suggested method is of $O(N^{3/2}\log^3 N)$.

3. Numerical Example

Consider a quasi-planar scatterer composed of short dipoles aligned in parallel to each other close to the *xy* plane. The dipoles are randomly spread on the plane with maximal height difference of h. Each dipole is loaded with a lumped inductor connected in the middle over an infinitesimally small gap. We assume, according to the thin wire approximation, that the current flows on the axis of the dipole while the boundary conditions are imposed on the cylindrical surface. The current on the dipole is assumed to be $\hat{z}$-directed with the maximum in the gap, decaying linearly to zero at the ends. Substituting all the field contributions into the boundary condition on the dipole surface and, applying it on all dipoles in the scatterer yields a MoM set of equations.

The proposed analysis applied to the described case is studied for error controllability as well as the stability of the algorithm. The interpolation precision as well as the truncation of the interacting basis is set to a certain threshold, 10^{-4}. Also the finest domain size is defined, 4×4 blocks. For these conditions the algorithm is tested on scatterers with increasingly larger number of unknowns and the normalized produced RMS error, of the attained current vector vs. the directly calculated one, is examined. With the increase of the scatterer area the number of levels in the multilevel process is increased, proportionally to the logarithm of the number of unknowns. It is therefore expected that for fixed thresholds the overall error of the algorithm would grow slightly due to the accumulation of errors from each level. Consequently, the resulting error becomes roughly proportional to the logarithm of the number of unknowns as seen in.

This growth can be mitigated by slightly tightening both the interpolation and the truncation tolerances, to preserve a constant error level while increasing the problem size. The first implies increasing the NG and PG densities or employing higher order interpolation rules, while the second suggests taking more radiating basis functions in the decomposition of the sub-domain currents.

4. References

1. E. Michielssen, A. Boag, and W. C. Chew, "Scattering from elongated objects: direct solution in O(N $\log^2$ N) operations", *IEE Proc.-Microw. Antennas Propagat.*, **143**, Aug. 1996, pp. 277-283.
2. R.J. Adams, A. Zhu, and F. X. Canning, "Efficient solution of integral equations in a localizing basis", *J. Elect. Mag. Wav. and App.*, **19**, May. 2005, pp. 1585-1594.
3. V. Rokhlin and P. G. Martinsson, "A fast direct solver for boundary integral equations in two dimensions", *J. Comp. Phys.*, **205**, 2005, pp. 1-23.
4. A. Boag, E. Michielssen, and A. Brandt, "Nonuniform polar grid algorithm for fast field evaluation", *IEEE Antennas Propagat. Lett.*, **1**, Oct. 2002, pp. 142-145.