Bandwidth Limitations in Small Antennas Composed of Negative Permittivity Materials and Metamaterials

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Abstract

This paper presents an overview of the bandwidth limitations of small antennas composed of negative permittivity materials and metamaterials. A quasi-static analysis is used to derive an equation for the *Q*-factor of an electrically small negative permittivity resonator of arbitrary shape, where the material composing the resonator is assumed to have an arbitrary dispersion relationship. The resulting equation depends upon the frequency derivative of the permittivity, the resonant permittivity value, and a geometrical shape factor. For spheres and ellipsoids, the resonant permittivity and shape factor are determined from well-known analytical results; in the case of arbitrary geometries, these parameters are determined numerically. The Landau-Lifshitz criteria on the frequency derivative of the permittivity [1] place a lower bound on the achievable *Q*-factor; in order to achieve this lower bound, the negative permittivity material must obey the Drude dispersion relationship. If we assume a Drude material, the equation for the *Q*-factor reduces to Wheeler's classic equation for the *Q*-factor of a small antenna [2]. We conclude that small negative permittivity antennas behave as elementary lumped element small antennas like those originally discussed by Wheeler. This conclusion is supported by recent studies illustrating the striking similarities in the resonant modes of the negative permittivity sphere and the spherical helix antenna. Finally, we present numerical simulations of antennas confirming that the quasi-static analysis of the *Q*-factor accurately predicts the antenna bandwidth performance.

1. Introduction

The application of negative electric permittivity to electrically small antennas has attracted attention recently [3, 4], due to the resonant properties of negative permittivity structures (which can resonate at sizes much smaller than a wavelength), and the possibility of a metamaterial implementation of such structures. It is important to recognize that negative permittivity resonators behave in a manner very similar to conventional conductor-based electrically small resonators (for example, normal mode helix structures and top-loaded dipoles), and are bound by the same fundamental limitations on *Q*-factor and bandwidth as all antenna structures (the Chu limit [5], which states that the *Q* of an antenna must be greater than $1/(ka)^3$ for $ka \ll 1$, where k is the wavenumber and a is the radius of the smallest sphere enclosing the antenna). This paper presents an analysis illustrating these limitations for the general case of arbitrary geometry and dispersion relationship, extending the work done in earlier studies of specific geometries and dispersion relationships.

2. Quasi-static analysis

We assume an electrically small structure composed of a material whose electric permittivity is negative over the frequencies of operation. Because we are considering geometries much smaller than a wavelength, electrostatics can be used to derive the basic properties of these structures. The *Q*-factor of an electrically small negative permittivity sphere obeying the Drude dispersion relationship was derived using a quasi-static analysis in reference [3]; here we generalize that analysis. We begin by assuming that the electrostatic polarizability α of the structure obeys the following form around its resonant value of permittivity:

$$\alpha = fV(\varepsilon - 1)/(\varepsilon - \varepsilon_{res}) \tag{1}$$

where ε is the permittivity of the structure, ε_{res} is the resonant value of the permittivity, V is the volume of the particle, and f is the shape factor, a scaling factor that depends upon the shape of the structure. The parameter ε_{res} is a negative number, meaning that the electrostatic polarizability is singular at some negative value of ε; this corresponds to the resonance of the structure. For a sphere, Eq. (1) assumes the well-known form where ε_{res}= -2 and f = 3; closed-form solutions for f and ε_{res} also exist for ellipsoids [6]. In the case of spherical shells, a closed-form solution exists for α [7], and this solution is very well approximated by Eq. (1) over a range of permittivities around the resonant value, enabling determination of f and ε_{res}. For arbitrary geometries, these parameters are determined numerically using electrostatics,

as follows: the dielectric structure is immersed in a uniform electric field and the resulting polarization is integrated over the volume of the structure; this simulation is performed over a range of permittivity values and the resulting α is fitted using Eq. (1) with f and ε_{res} as fitting parameters.

For a given dispersion relationship, Eq. (1) predicts reasonably well the actual resonant frequency of electrically small structures. Although the electrostatic polarizability is singular at resonance, radiative damping, which is not accounted for in the electrostatic calculation, insures that the polarizability actually remains finite. The Q-factor of the resonance can be computed by accounting for radiative damping. The induced polarization in the particle introduces a radiation reaction field, yielding an effective polarizability of the particle [3]:

$$\alpha_{eff} = \alpha \Big/ \left(1 - i\frac{\alpha}{4\pi}\frac{2k^3}{3}\right) \tag{2}$$

where k is the wavenumber. Following the derivation in [3], the magnitude-squared of the effective polarizability can be written as a Lorentzian about its resonant frequency, and the Q-factor is derived directly from this function. The analysis is applied to arbitrary dispersion relationships by expanding the permittivity in a Taylor series about its resonant value:

$$\varepsilon(\omega) = \varepsilon(\omega_o) + \varepsilon'(\omega - \omega_o) \tag{3}$$

where ω is the angular frequency, ω_o the resonant frequency, and ε' is the frequency derivative of the permittivity evaluated at the resonant frequency. Combining Eqs. (1)-(3), we obtain the following equation for the Q-factor of the negative permittivity resonator:

$$Q = \left(3\pi\omega_o\varepsilon'\right) \Big/ \left(fV\left[1 - \varepsilon_{res}\right]k^3\right). \tag{4}$$

Eq. (4) can be simplified further by assuming the material obeys the Drude dispersion relationship, $\varepsilon = 1 - \omega_p^2 / \omega^2$, where ω_p is the plasma frequency. This yields the following equation for the Q-factor:

$$Q_{Drude} = \frac{6\pi}{fVk^3}. \tag{5}$$

3. Discussion

In a small antenna, it is desirable to minimize the radiation Q, as this yields the widest bandwidth of operation. Eq. (4) indicates that the Q can be minimized by minimizing the frequency derivative of the permittivity. However, materials with negative permittivity are necessarily dispersive: in order to satisfy causality, the frequency derivative of the permittivity must obey two well-known inequalities [1, 8]. The more restrictive of these two inequalities for the case of negative values of permittivity is the following:

$$\varepsilon' \geq 2(1 - \varepsilon)/\omega \tag{6}$$

It is easily verified that the Drude dispersion relationship achieves the minimum value defined by Eq. (6) at all frequencies. Therefore, the Q-factor of a negative permittivity resonator is minimized by choosing a material with the Drude dispersion relationship. Dispersion 'engineering' cannot improve the Q-factor beyond that achieved by a Drude material without violating causality, and furthermore, a metamaterial implementation of a negative permittivity material will achieve this Q only if it obeys the Drude dispersion relationship. The result in Eq. (5) thus represents the optimal Q achievable by a negative permittivity resonator. In this case, the Q-factor is a simple function of the volume of the antenna (normalized to the wavelength cubed) and the shape factor f. Eq. (5) is, in fact, identical to Wheeler's formula for the Q of a lumped-element small antenna [2, 9].

We consider this result for a few simple geometries. For the case of a sphere, f=3, V=4/3πa^3 (where a is the radius), giving Q=(3/2)$(ka)^{-3}$. This is the result derived in [3]: the Q of the negative permittivity sphere with a Drude dispersion relationship is 1.5x the Chu limit [5]. Similar values of Q are obtained in small spherical antenna designs such as the spherical helix [10] and spherical capped dipole [9]. An antenna composed of a negative permittivity sphere with a Drude dispersion relationship will exhibit performance comparable to these conventional spherical antennas of the same normalized volume. A detailed study of the resonant modes of the tapered spherical helix and the negative permittivity sphere reveals many similarities in the behavior of these two structures [11].

A hollow spherical shell of negative permittivity material [4] achieves a Q nearly equal to that of the solid sphere; in this case the shape factor f varies between 2.8 and 3, depending upon the inner radius of the shell relative to the outer radius. The smaller value of f yields a slightly larger Q-factor for the spherical shell as compared to that of a solid sphere with the same normalized volume. In the limits of small inner radius (nearly solid) and large inner radius (very thin shell), the Q approaches that of the solid sphere.

The resonant permittivities and shape factors for ellipsoids (determined from closed form solutions [6]) and cylinders (determined numerically) for various aspect ratios are shown in Figs. 1(a)-(b). Also shown in Fig. 1(b) is Wheeler's shape factor for the air-filled cylindrical plate capacitor (used to determine the Q of a capped dipole antenna). For aspect ratios (height/width) less than 1.5, the Q-factor of a negative permittivity cylinder is larger than that of a cylindrical plate capacitor (due to the smaller f); in the opposite extreme (large aspect ratios) the Q of the negative permittivity structure is lower than that of the capacitor. Over a wide range of moderate aspect ratios the Q-factors of the two structures are *not* dramatically different. The negative permittivity cylinder antenna (with a Drude dispersion relationship) will exhibit performance similar to a small capped dipole antenna occupying the same normalized volume.

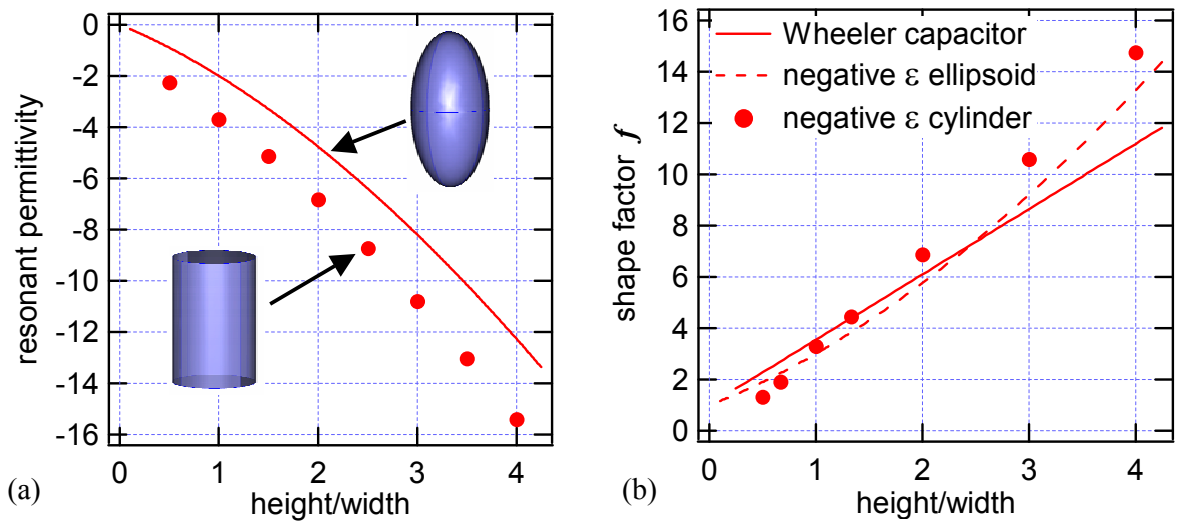

Figure 1. (a) Resonant permittivity value vs. aspect ratio for ellipsoids (line) and cylinders (circles), as determined using electrostatics. The resonance is assumed to be polarized along the 'height' direction (vertically). The actual resonant frequency of the structure approaches the electrostatic value in the limit of very small electrical size, and will shift by 5-10% as the size increases. (b) The shape factor f for the negative permittivity ellipsoid and cylinder vs. aspect ratio. Also shown is the shape factor of the Wheeler capacitor. The Q is derived from f using Eq. (5). The Q-factor derived by the quasi-static analysis very accurately predicts the actual Q of an antenna formed from the resonator.

4. Antennas

In order to illustrate the effectiveness of the quasi-static analysis in predicting the antenna Q, we consider antennas composed of ellipsoids with an aspect ratio of 2 (the full dipole height is twice the width). A vertically polarized monopole antenna is created by feeding a half ellipsoid on an infinite ground plane with a small stub inserted at its center (see inset, Fig. 2). The antenna is fed with a 50 ohm coaxial cable, and is impedance-matched simply by varying the length of the stub to optimize the match, in the same manner as was described for the case of a sphere in [3]. In Fig. 2, the solid line is the Q-factor vs. ka predicted from the quasi-static analysis. Antennas with varying ka are generated by holding the physical size constant and varying the plasma frequency of the negative permittivity material (a Drude dispersion relationship is assumed). The Q-factor is determined at the matched frequency from the impedance data according to the equations derived in [8]. These values are plots as the circles in the plot. This figure illustrates that the quasi-static analysis used to analyze the resonators gives a very accurate prediction of the Q-factor of the antenna formed by these resonators. Good accuracy was also observed in the spherical antenna discussed in [3], and is also observed for cylindrical structures.

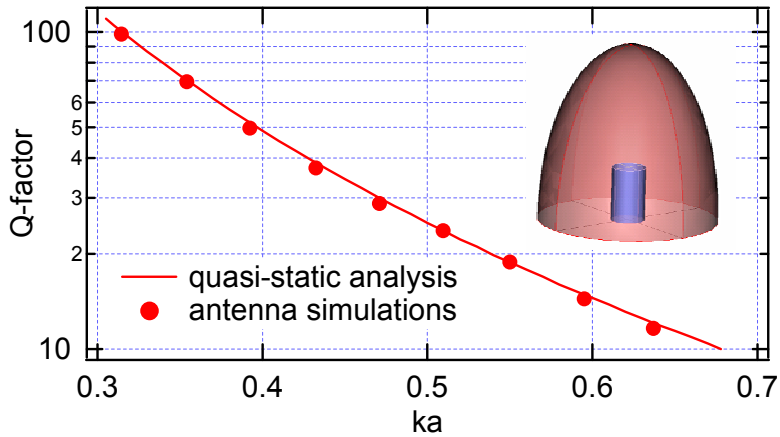

Figure 2. *Q*-factor vs. *ka* for a negative permittivity (Drude) ellipsoid antenna with an aspect ratio of 2 (aspect ratio is defined in terms of the height of the full ellipsoid). The solid line shows the *Q* derived from the quasi-static analysis of the resonator (Eq. 5). The circles are derived from a full harmonic simulation of the antenna structure: for each point, the antenna is impedance-matched to 50 ohms (by simply adjusting the length of the stub), and the *Q*-factor is determined at the matched frequency from the impedance behavior. The quasi-static analysis accurately predicts the *Q* over a wide range of normalized size *ka*.

5. Conclusion

The bandwidth performance of a negative permittivity antenna depends upon the normalized volume of the antenna, its shape, and the dispersion relationship of the material. A Drude material yields the optimal *Q*-factor; in this case the antenna behaves in a manner similar to Wheeler's lumped element small antenna. Negative electric permittivity can be achieved using plasmas, and is also observed in certain metamaterial structures. It is important to emphasize that the electromagnetic behavior and performance observed in these antennas will, at best, match that seen in other more conventional electrically small antenna structures of the same normalized size. The study of negative permittivity antennas provides a useful framework for demonstrating the universality of the limitations on small antennas.

6. References

[1] L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media*, Addison-Wesley Reading, MA, 1960.

[2] H.A. Wheeler, "Fundamental limitations of small antennas," *Proc. IRE,* vol. 35, pp. 1479-1484, 1947.

[3] H.R. Stuart and A. Pidwerbetsky, "Electrically small antenna elements using negative permittivity resonators," *IEEE Trans. Ant. Prop.,* vol. 54, pp. 1644-1653, 2006.

[4] R.W. Ziolkowski and A. Erentok, "Metamaterial-based efficient electrically small antennas," *IEEE Trans. Ant. Prop.,* vol. 54, pp. 2113-2130, 2006.

[5] L.J. Chu, "Physical limitations on omni-directional antennas," *J. Appl. Phys.,* vol. 19, pp. 1163-1175, 1948.

[6] C.F. Bohren and D.R. Huffman, *Absorption and scattering of light by small particles*, John Wiley & Sons, Inc., New York, 1983.

[7] R.D. Averitt, S.L. Westcott, and N.J. Halas, "Linear optical properties of gold nanoshells," *JOSA B,* vol. 16, pp. 1824-1832, 1999.

[8] A.D. Yaghjian and S.R. Best, "Impedance, bandwidth, and *Q* of antennas," *IEEE Trans. Ant. Prop.,* vol. 53, pp. 1298-1324, 2005.

[9] A.R. Lopez, "Fundamental limitations of small antennas: Validation of Wheeler's formulas," *IEEE Ant. Prop. Mag.,* vol. 48, pp. 28-36, 2006.

[10] S.R. Best, "The radiation properties of electrically small folded spherical helix antennas," *IEEE Trans. Ant. Prop.,* vol. 52, pp. 953-960, 2004.

[11] H.R. Stuart, "An electromagnetic comparison of the tapered spherical helix and the negative permittivity sphere," in *IEEE Int. Symp. on Antennas and Propagation*, Honolulu, HI, 2007, pp. 3472 - 3475.