Lower Bounds on Quality Factor for Antennas Containing Bianisotropic Material

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Abstract

With the possibility of new metamaterials having constitutive parameters with unusual values, such as permittivity and permeability with negative real parts, it is conceivable that these metamaterials could be used to increase the bandwidth of antennas, especially electrically small antennas. However, in this paper we show that a quality factor (Q) that appears to be generally applicable to a single resonance or antiresonance of a tuned antenna containing lossy or lossless dispersive bianisotropic material has a spherical lower bound that cannot be less than the least Chu lower bound, which occurs for antennas that have their radiated power divided equally between electric and magnetic spherical multipoles. For these antennas, it is further shown that the least lower bound can always be obtained, in principle, with electric and magnetic surface currents determined from the electromagnetic field equivalence formulas.

1. Introduction

The purpose of this paper is to determine the lower bounds on the quality factor (Q) of an antenna containing the most general linear, spatially nondispersive material, namely, lossy or lossless, inhomogeneous, temporally dispersive, bianisotropic material. In particular, we ask whether the spherical lower bounds on Q determined by Chu and others [1]–[5] remain valid for antennas containing material (or metamaterial) that obey the following general constitutive relations (for fields with $e^{j\omega t}$, $\omega > 0$ time dependence)

$$\mathbf{B}(\mathbf{r},\omega) = \overline{\boldsymbol{\mu}}(\mathbf{r},\omega)\cdot\mathbf{H}(\mathbf{r},\omega) + \overline{\boldsymbol{\nu}}(\mathbf{r},\omega)\cdot\mathbf{E}(\mathbf{r},\omega)\,, \quad \mathbf{D}(\mathbf{r},\omega) = \overline{\boldsymbol{\epsilon}}(\mathbf{r},\omega)\cdot\mathbf{E}(\mathbf{r},\omega) + \overline{\boldsymbol{\tau}}(\mathbf{r},\omega)\cdot\mathbf{H}(\mathbf{r},\omega) \tag{1}$$

where $\mathbf{B}$ is the magnetic induction, $\mathbf{H}$ is the magnetic field, $\mathbf{E}$ is the electric field, $\mathbf{D}$ is the electric displacement, the $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\epsilon}}$ are the complex permeability and permittivity dyadics, respectively, and $\overline{\boldsymbol{\nu}}$ and $\overline{\boldsymbol{\tau}}$ are the complex magnetoelectric dyadics.

2. Quality Factor of Antennas with Bianisotropic Material

To help answer this question, we begin with an expression for Q that was found in [6], [7] to maintain the accuracy of the proportionality between Q and twice the inverse of the matched VSWR half-power fractional bandwidth for antennas containing highly lossy dispersive bianisotropic material; specifically

$$Q(\omega_0) = \omega_0 W^{\mathrm{Q}}(\omega_0)/P_A(\omega_0) \tag{2}$$

for a sufficiently isolated resonance or antiresonance with $Q >> 1$ ($Q \gtrsim 2$ often suffices), where $P_A(\omega_0)$ is the power accepted (power radiated plus power loss) by the antenna and the "Q-energy", $W^{\mathrm{Q}}(\omega_0)$, is given by

$$W^{\mathrm{Q}}(\omega_0) = \frac{1}{4}\lim_{r\to\infty}\left[\int_{\mathcal{V}_o(r)}\Big|\,|\mathbf{H}^*\cdot(\omega_0\overline{\boldsymbol{\mu}})'\cdot\mathbf{H}| + |\mathbf{E}^*\cdot(\omega_0\overline{\boldsymbol{\epsilon}})'\cdot\mathbf{E}| + [\mathbf{E}^*\cdot(\omega_0\overline{\boldsymbol{\tau}})'\cdot\mathbf{H} + \mathbf{H}^*\cdot(\omega_0\overline{\boldsymbol{\nu}})'\cdot\mathbf{E}]\,\Big|\,d\mathcal{V} - 2\epsilon_0 r\int_{4\pi}|\mathbf{F}|^2 d\Omega\right]. \tag{3}$$

For isotropic materials (3) reduces to $W^{\mathrm{Q}}(\omega_0) = W_m^{\mathrm{Q}}(\omega_0) + W_e^{\mathrm{Q}}(\omega_0)$ with

$$W_m^{\mathrm{Q}} = \frac{1}{4}\lim_{r\to\infty}\left[\int_{\mathcal{V}_o(r)}|(\omega_0\mu)'|\,|\mathbf{H}|^2 d\mathcal{V} - \epsilon_0 r\int_{4\pi}|\mathbf{F}|^2 d\Omega\right], \quad W_e^{\mathrm{Q}} = \frac{1}{4}\lim_{r\to\infty}\left[\int_{\mathcal{V}_o(r)}|(\omega_0\epsilon)'|\,|\mathbf{E}|^2 d\mathcal{V} - \epsilon_0 r\int_{4\pi}|\mathbf{F}|^2 d\Omega\right] \tag{4}$$

and ϵ_0 is the permittivity of the free space in which the antenna system is assumed to reside. Unlike the internal energy within lossless (and nondispersive conductive) material, the "Q-energy" denoted by $W^{\mathrm{Q}}(\omega_0)$ cannot be related directly to the sum of kinetic, potential, and store electromagnetic field energy in highly lossy dispersive material [7]. These expressions in (2)–(4) apply to one-port, linear, passive antennas tuned at a frequency ω_0 (so that the input reactance $X_0(\omega_0) = 0$) to resonance ($X_0'(\omega_0) > 0$) or antiresonance ($X_0'(\omega_0) < 0$) [8], where primes denote differentiation with respect to the angular frequency ω. As shown in Fig. 1, $\mathcal{V}_a$ is the volume of the antenna material that lies outside the shielded power supply and reference plane S_0 in the feed line of the

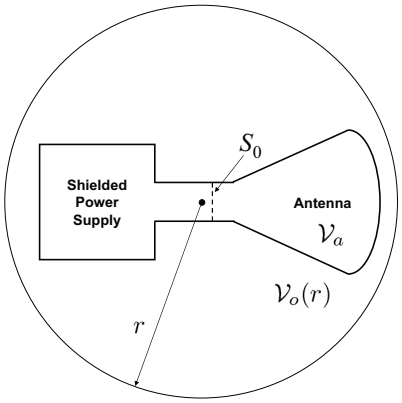

Figure 1: One-port, linear, passive antenna with feed and shielded power supply.

antenna system. Any capacitor or inductor used to tune the antenna is included in $\mathcal{V}_a$. The volume $\mathcal{V}_o(r)$, which includes the volume $\mathcal{V}_a$, is the entire volume outside the shielded power supply and reference plane S_0 out to a large sphere in free space of radius r that surrounds the antenna system (antenna plus shielded power supply). As $r \to \infty$, the volume $\mathcal{V}_o(r)$ becomes infinite. The solid angle integration element is $d\Omega = dS/r^2 = \sin\theta d\theta d\phi$ with (r, θ, ϕ) being the usual spherical coordinates of the position vector $\mathbf{r}$, and the complex far electric field pattern $\mathbf{F}(\theta, \phi)$ is defined by $\mathbf{F}(\theta, \phi) = \lim_{r\to\infty} re^{jkr}\mathbf{E}(\mathbf{r})$ with $k = \omega/c$ and c the speed of light in free space.

3. Lower Bounds on Quality Factor

The Q in (2) satisfies the exact relationship

$$Q(\omega_0) = \eta[Q_e(\omega_0) + Q_m(\omega_0) + Q^{\text{in}}(\omega_0) - k_0 a], \quad k_0 = \omega_0/c \tag{5}$$

which can be derived similarly to the corresponding relationship in [8, app. C]. The symbol η denotes the radiation efficiency of the antenna. The Q_e and Q_m are defined in terms of the energies stored in the reactive electric and magnetic fields (total minus radiation-field energy) in the free-space region outside the minimum radius a of the sphere that circumscribes the antenna system (antenna plus shielded power supply). (The origin of the coordinate system is now assumed to be the center of this circumscribing sphere.) The Q_e and Q_m are given explicitly by [4], [8, app. C, eq. (C.5)]

$$Q_e(\omega_0) = \frac{k_0}{2} \frac{\lim\limits_{r\to\infty}\Big[\int\limits_a^r \int\limits_{4\pi} |\mathbf{E}|^2 r^2 d\Omega\, dr - (r-a)\int\limits_{4\pi} |\mathbf{F}|^2 d\Omega\Big]}{\int\limits_{4\pi} |\mathbf{F}|^2 d\Omega} = \frac{k_0}{2} \frac{\int\limits_a^\infty dr \int\limits_{4\pi} (r^2|\mathbf{E}|^2 - |\mathbf{F}|^2) d\Omega}{\int\limits_{4\pi} |\mathbf{F}|^2 d\Omega} \tag{6a}$$

$$Q_m(\omega_0) = \frac{k_0}{2} \frac{\lim\limits_{r\to\infty}\Big[\int\limits_a^r \int\limits_{4\pi} \frac{\mu_0}{\epsilon_0}|\mathbf{H}|^2 r^2 d\Omega\, dr - (r-a)\int\limits_{4\pi} |\mathbf{F}|^2 d\Omega\Big]}{\int\limits_{4\pi} |\mathbf{F}|^2 d\Omega} = \frac{k_0}{2} \frac{\int\limits_a^\infty dr \int\limits_{4\pi} (r^2|\mathbf{H}|^2 - |\mathbf{F}|^2) d\Omega}{\int\limits_{4\pi} |\mathbf{F}|^2 d\Omega}\,. \tag{6b}$$

In [8, app. C], it is proven that $Q_e > 0$ and $Q_m > 0$. The contribution Q^{in} to the quality factor from inside the circumscribing sphere of radius a is

$$Q^{\text{in}}(\omega_0) = \frac{k_0}{2\epsilon_0} \frac{\int\limits_{\mathcal{V}_o(a)} \Big| |\mathbf{E}^* \cdot (\omega_0 \overline{\boldsymbol{\epsilon}})' \cdot \mathbf{E}| + |\mathbf{H}^* \cdot (\omega_0 \overline{\boldsymbol{\mu}})' \cdot \mathbf{H}| + [\mathbf{E}^* \cdot (\omega_0 \overline{\boldsymbol{\tau}})' \cdot \mathbf{H} + \mathbf{H}^* \cdot (\omega_0 \overline{\boldsymbol{\nu}})' \cdot \mathbf{E}] \Big| d\mathcal{V}}{\int\limits_{4\pi} |\mathbf{F}|^2 d\Omega} \geq 0\,. \tag{7}$$

The antenna is assumed to be resonant at the frequency ω_0, that is, it's input reactance is zero either naturally or because of a series tuning inductor or capacitor (considered part of the volume $\mathcal{V}_a$ of antenna material). The input reactance of an antenna can be written explicitly in terms of the fields of the antenna as [8, eq. (53)]

$$X_0(\omega_0) = \frac{\omega_0}{|I_0(\omega_0)|^2} \text{Re} \int\limits_{\mathcal{V}_o} [\mathbf{H} \cdot \overline{\boldsymbol{\mu}}^* \cdot \mathbf{H}^* - \mathbf{E} \cdot \overline{\boldsymbol{\epsilon}}^* \cdot \mathbf{E}^* + \mathbf{H} \cdot (\overline{\boldsymbol{\nu}}^* - \overline{\boldsymbol{\tau}}_t) \cdot \mathbf{E}^*]\, d\mathcal{V} = 0\,. \tag{8}$$

Generally, $Q_e(\omega_0) + Q_m(\omega_0) \gg k_0 a$ and (5) can be well approximated by

$$Q(\omega_0) \approx \eta[Q_e(\omega_0) + Q_m(\omega_0) + Q^{\text{in}}(\omega_0)]\,. \tag{9}$$

Since $Q_e(\omega_0)$, $Q_m(\omega_0)$, and $Q^{\text{in}}(\omega_0)$ are all greater than or equal to zero, the minimum Q in (9) would occur for a given $Q_e(\omega_0)$ and $Q_m(\omega_0)$ if $Q^{\text{in}}(\omega_0) = 0$. In the previous determinations of the lower bounds on

quality factor [1]–[5], it was assumed that $Q^{\rm in}(\omega_0)$ could be zero only if the fields of the antenna inside the circumscribing sphere of radius a, that is, the $\mathbf{E}$ and $\mathbf{H}$ fields in (7), were zero. But then (8) and (6) would imply that $X_0(\omega_0)$ could not be zero unless $Q_e(\omega_0) = Q_m(\omega_0)$. Thus, in general, it was assumed that $Q^{\rm in}(\omega_0)$ could not be zero but had to be at least as large as the quality factor of a tuning inductor or capacitor that would make $Q^{\rm in}(\omega_0) = |Q_e(\omega_0) - Q_m(\omega_0)|$, and thus the spherical lower bound on quality factor (often called the Chu lower bound) was assumed to be

$$Q_{\rm lb}^{\rm Chu}(\omega_0) = \eta {\rm Max}[2Q_e(\omega_0), 2Q_m(\omega_0)]\,. \tag{10}$$

For lossy dispersive materials, however, it is presumably possible for $(\omega_0\overline{\boldsymbol{\epsilon}})'$, $(\omega_0\overline{\boldsymbol{\mu}})'$, and $[(\omega_0\overline{\boldsymbol{\nu}})', (\omega_0\overline{\boldsymbol{\tau}})']$ to all be zero in (7), and thus $Q^{\rm in}(\omega_0) = 0$, without $\mathbf{E}$ and $\mathbf{H}$ being zero inside the sphere of radius a. That is, $Q^{\rm in}(\omega_0)$ can be zero and yet there can be contributions from inside the sphere of radius a to the input reactance (8) of the antenna that allow the antenna to be tuned (have zero reactance at ω_0) even though $Q_m(\omega_0)$ may not equal $Q_e(\omega_0)$. Thus, for antennas that contain lossy dispersive materials, the minimum possible value (lower bound) for the Q in (2) is simply

$$Q_{\rm lb}(\omega_0) = \eta[Q_e(\omega_0) + Q_m(\omega_0) - k_0 a] \overset{Q \gg k_0 a}{\approx} \eta[Q_e(\omega_0) + Q_m(\omega_0)]\,. \tag{11}$$

Writing $Q_m = \gamma Q_e$ $(0 \le \gamma \le 1)$ if $Q_m \le Q_e$, or $Q_e = \gamma Q_m$ $(0 \le \gamma \le 1)$ if $Q_e \le Q_m$, we see from (10) and (11) that the lower bound on the improved Q in (2) is related to the Chu lower bound by

$$Q_{\rm lb}(\omega_0) \approx \frac{1+\gamma}{2} Q_{\rm lb}^{\rm Chu}(\omega_0) = \eta \frac{1+\gamma}{2} {\rm Max}[2Q_e(\omega_0), 2Q_m(\omega_0)] \tag{12}$$

where $\eta < 1$ in (11) and (12) because the antenna material inside the sphere of radius a has to be lossy for (11) and (12) to hold. For electric and magnetic dipole fields outside the circumscribing sphere of radius a and $k_o a \ll 1$, we have

$$Q_{\rm lb}^{\rm Chu}(\omega_0) \approx \eta \frac{1}{(1+\gamma)(k_0 a)^3} \tag{13}$$

since the stored energy in electric and magnetic dipoles are predominantly in the electric and magnetic fields, respectively, whereas the power radiated by the dipoles depends equally on the values of Q_e and Q_m. (For example, $Q_{\rm lb}^{\rm Chu}(\omega_0) \approx \eta(k_0 a)^{-3}$ if $Q_e(\omega_0) = 0$ or $Q_m(\omega_0) = 0$, and $Q_{\rm lb}^{\rm Chu}(\omega_0) \approx \eta(k_0 a)^{-3}/2$ if $Q_e(\omega_0) = Q_m(\omega_0)$ [1]–[5].) Equations (12) and (13) give

$$Q_{\rm lb}(\omega_0) \approx \eta \frac{1}{2(k_0 a)^3}, \quad k_0 a \ll 1\,. \tag{14}$$

The least $Q_{\rm lb}^{\rm Chu}(\omega_0)$ for dipole antennas (or any other antenna) occurs if $\gamma = 1$, in which case (12) shows that

$$Q_{\rm lb}(\omega_0) \approx Q_{\rm lb}^{\rm Chu}(\omega_0)\,. \tag{15}$$

For tuned antennas that are lossless ($\eta = 1$) in a frequency window about ω_0, the inequality

$${\rm Re}\left\{\mathbf{H}^* \cdot (\omega\overline{\boldsymbol{\mu}})' \cdot \mathbf{H} + \mathbf{E}^* \cdot (\omega\overline{\boldsymbol{\epsilon}})' \cdot \mathbf{E} + \mathbf{E} \cdot [(\omega(\overline{\boldsymbol{\nu}}_t + \overline{\boldsymbol{\tau}}^*)]' \cdot \mathbf{H}^*\right\} \ge \left[\mu_0 |\mathbf{H}|^2 + \epsilon_0 |\mathbf{E}|^2\right] \tag{16}$$

obtained in [8, app. B] combines with (7) to show that

$$Q^{\rm in}(\omega_0) \ge \frac{k_0}{2} \frac{\int\limits_{\mathcal{V}_o(a)} \left[\frac{\mu_0}{\epsilon_0}|\mathbf{H}|^2 + |\mathbf{E}|^2\right] d\mathcal{V}}{\int\limits_{4\pi} |\mathbf{F}|^2 d\Omega} \ge 0\,. \tag{17}$$

From (5) and (17) we see that the lower bound on Q is again given by (11) if $Q^{\rm in}(\omega_0) = 0$, which can now occur only if the antenna fields are zero inside the sphere of radius a. Since the antenna is tuned ($X_0 = 0$), however, and the antenna fields are zero inside the radius a, all the input reactance of the antenna comes from the antenna fields that lie outside the sphere of radius a. Thus, from (6) and (8), the lower bound in (11) can only be perfectly realized for lossless antennas if $Q_e = Q_m$. On the other hand, one could argue that it may be possible for $Q^{\rm in}$ to be negligible if the integral in the numerator of (17) is negligible while the material constitutive parameters are so large in the antenna material that the input reactance in (8) can be made zero. It will now be shown, however, that this is impossible, at least for antennas that contain only lossless isotropic materials (scalar ϵ and μ with $\overline{\boldsymbol{\tau}}$ and $\overline{\boldsymbol{\nu}}$ zero).

For lossless isotropic materials and $Q_e \ne Q_m$, it is not possible to make the input reactance in (8) zero while keeping $Q^{\rm in}$ in (7) negligible unless $|\epsilon| > (\omega_0\epsilon)'$ and $|\mu| > (\omega_0\mu)'$. But it is easily proven [7] that these two inequalities are incompatible with the inequalities [8], [9, eqs. (84.1) and (84.2)]

$$(\omega\epsilon)' - \epsilon_0 \ge \frac{\omega\epsilon'}{2} \ge 0\,, \quad (\omega\mu)' - \mu_0 \ge \frac{\omega\mu'}{2} \ge 0 \tag{18}$$

for lossless isotropic materials. Thus, for antennas that contain only lossless isotropic material, the internal energy in Q^{in} must be large enough to tune the antenna, and consequently the lower bound on Q approximately equals the Chu lower bound; that is

$$Q_{\text{lb}}(\omega_0) = Q_{\text{lb}}^{\text{Chu}}(\omega_0) - k_0 a \overset{Q \gg k_0 a}{\approx} Q_{\text{lb}}^{\text{Chu}}(\omega_0) = \text{Max}[2Q_e(\omega_0), 2Q_m(\omega_0)]\,. \tag{19}$$

For electric and magnetic dipole fields outside the circumscribing sphere of radius a, (19) combines with (13) (with $\eta = 1$) to give

$$Q_{\text{lb}}(\omega_0) \approx \frac{1}{(1+\gamma)(k_0 a)^3}, \quad k_0 a \ll 1\,. \tag{20}$$

The expressions in (11)–(12) and (19) give the lower bounds on the improved Q of (2) for lossy dispersive antennas and lossless dispersive antennas (containing only isotropic materials), respectively. To achieve the lossy lower bound in (11)–(12) for $\gamma \neq 1$, efficiency would have to be sacrificed — and to what extent, especially for $\gamma \ll 1$, is not known. Moreover, as noted above in (15), the least possible lower bound for an antenna is always equal to the least possible Chu lower bound, which occurs if $\gamma = 1$.

3.1 Attaining the Least Lower Bound

For antennas with $Q_e = Q_m$ ($\gamma = 1$), the least lower bound on Q is reached if the fields inside the circumscribing sphere of radius a are zero. This least quality factor can, in principle, be attained for the antenna by exciting these fields with electric and magnetic surface currents, $\mathbf{K}_e$ and $\mathbf{K}_m$, on the surface of this sphere given by

$$\mathbf{K}_e = \hat{\mathbf{n}} \times \mathbf{H}\,, \qquad \mathbf{K}_m = -\hat{\mathbf{n}} \times \mathbf{E} \tag{21}$$

where $\hat{\mathbf{n}}$ is the outward normal to the sphere [10]. The Stratton-Chu equivalence formulas [11, sec. 8.14] ensure that the fields produced by these surface currents will be zero inside the sphere (a result sometimes referred to as the "extinction theorem").

The lower bounds on Q in (11)–(12), (15), and (19) hold for all one-port, linear, passive, antennas containing the most general spatially nondispersive lossy material (or lossless isotropic material in (19)) whose real parts of permittivity and permeability can be either negative or positive (or zero). Of course, if nonlinear or active material is allowed in the antenna proper or in its tuning elements, or if multiple resonances and antiresonances closely overlap, then twice the inverse of the matched VSWR half-power fractional bandwidth is not limited by these lower bounds on Q.

4. References

[1] L.J. Chu, "Physical limitations of omni-directional antennas," *J. Appl. Phys.*, **19**,1948, pp. 1163–1175.

[2] R.F. Harrington, *Time-Harmonic Electromagnetic Fields*, New York: McGraw-Hill, 1961, sec. 6-13.

[3] R.E. Collin, S. Rothschild, "Evaluation of antenna Q," *IEEE Trans. Ant. Prop.*, **AP-12**, 1964, pp. 23–27.

[4] R.L. Fante, "Quality factor of general ideal antennas," *IEEE Trans. Ant. Prop.*, **AP-17**, 1969, pp. 151–155.

[5] J.S. McLean, "A re-examination of the fundamental limits on the radiation Q of electrically small antennas," *IEEE Trans. Antennas Propagat.*, **44**, 1996, pp. 672–676.

[6] A.D. Yaghjian, "Improved formulas for the Q of antennas with highly lossy dispersive materials," *IEEE Antennas Wireless Propagat. Lett.*, **5**, 2006, pp. 365–369.

[7] A.D. Yaghjian, "Internal energy, Q-energy, Poynting's theorem, and the stress dyadic in dispersive material," *IEEE Trans. Antennas Propagat.*, **55**, 2007, pp. 1495–1505.

[8] A.D. Yaghjian and S.R. Best, "Impedance, bandwidth, and Q of antennas," *IEEE Trans. Antennas Propagat.*, **53**, 2005, pp. 1298–1324.

[9] L.D. Landau, E.M. Lifshitz, and L.P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd Edition, Oxford: Butterworth-Heinemann, 1984.

[10] H.A. Wheeler, "Small Antennas," Sec. 6-5 in *Antenna Engineering Handbook, 2nd Edition*, Eds. R.C. Johnson and H. Jasik, New York: McGraw-Hill, 1984.

[11] J.A. Stratton, *Electromagnetic Theory*, New York: McGraw-Hill, 1941.