Fields in Perfect Electromagnetic Conductor

Ismo V. Lindell
Department of Radio Science and Engineering, Helsinki University of Technology
Box 3000, Espoo 02015, Finland
ismo.lindell@tkk.fi

Abstract

The problem of finding unique fields in perfect electromagnetic conductor (PEMC) is considered by analyzing transmission of an obliquely incident plane wave to PEMC half space and slab. It is shown that uniqueness depends on the exact definition of the PEMC medium conditions. Three possible medium conditions are tested, out of which only one, defining PEMC as the limiting case of a Tellegen medium, leads to uniqueness for fields inside the PEMC. Reflected fields are unique and the fields transmitted through the PEMC slab vanish for all cases.

1. Introduction

The perfect electromagnetic conductor (PEMC) has been defined by the medium conditions [1,2]

$$\mathbf{D} = M\mathbf{B}, \quad \mathbf{H} = -M\mathbf{E}, \tag{1}$$

where M is the PEMC admittance parameter, or axion parameter [3]. In this study, the medium is assumed lossless, whence M is restricted to have real values, positive or negative. The classical perfect electric conductor (PEC) and perfect magnetic conductor (PMC) are two special cases corresponding to the respective parameter values $1/M = 0$ and $M = 0$.

PEMC can be also defined as a special case of Tellegen medium [4] obeying the medium conditions of the form [1]

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = q \begin{pmatrix} M & 1 \\ 1 & 1/M \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \tag{2}$$

in the limit of the parameter $|q| \to \infty$. In fact, (2) can be expressed in equivalent form as $\mathbf{D} - M\mathbf{B} = 0$ and $\mathbf{H} + M\mathbf{E} = \mathbf{D}/q$, As another similar representation we can write

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = q \begin{pmatrix} 1/M & -1 \\ -1 & M \end{pmatrix} \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}, \tag{3}$$

for $|q| \to \infty$. This corresponds to the conditions $\mathbf{H} + M\mathbf{E} = 0$ and $\mathbf{D} - M\mathbf{B} = -\mathbf{H}/q$. Of course, (3) is not just (2) inverted, since the matrices in these two expressions have no inverses.

To see the difference of these definitions, we analyze the problem of fields transmitted to a PEMC half space and slab for an obliquely incident plane wave. The problem of interior fields in PEMC was studied in [5] for normally incident waves with the result that the transmitted fields could not be uniquely determined when the PEMC conditions (1) were used. In the present study we compare the three PEMC medium conditions to find uniqueness.

2. Plane-wave fields

Let us consider the classical problem of a plane wave incident in the half space $z > 0$ to a boundary interface at $z = 0$ and causing a reflected wave. The electric and magnetic fields are assumed to have the form

$$\mathbf{E}^i(\mathbf{r}) = \mathbf{E}^i \exp(-j\mathbf{k}^i \cdot \mathbf{r}), \quad \mathbf{H}^i(\mathbf{r}) = \mathbf{H}^i \exp(-j\mathbf{k}^i \cdot \mathbf{r}), \tag{4}$$

$$\mathbf{E}^r(\mathbf{r}) = \mathbf{E}^r \exp(-j\mathbf{k}^r \cdot \mathbf{r}), \quad \mathbf{H}^r(\mathbf{r}) = \mathbf{H}^r \exp(-j\mathbf{k}^r \cdot \mathbf{r}), \tag{5}$$

where the two wave vectors are defined by

$$\mathbf{k}^i = -k_z\mathbf{u}_z + k_x\mathbf{u}_x, \quad \mathbf{k}^r = k_z\mathbf{u}_z + k_x\mathbf{u}_x, \tag{6}$$

and k_x is assumed to be a positive real number. The half space $z > 0$ is assumed empty, whence the wave-vector components satisfy $k_z^2 + k_x^2 = k_o^2$

From the Maxwell equations relations for the fields transverse to the axial direction can be found as

$$\mathbf{H}_t^i = \overline{\overline{\mathsf{Y}}}_t \cdot \mathbf{E}_t^i, \qquad \mathbf{H}_t^r = -\overline{\overline{\mathsf{Y}}}_t \cdot \mathbf{E}_t^r. \tag{7}$$

$\overline{\overline{\mathsf{Y}}}_t$ is the free-space admittance dyadic. It can be expressed in terms of a dimensionless dyadic $\overline{\overline{\mathsf{J}}}_t$

$$\overline{\overline{\mathsf{Y}}}_t = \frac{1}{\eta_o}\overline{\overline{\mathsf{J}}}_t, \quad \overline{\overline{\mathsf{J}}}_t = \frac{1}{k_o k_z}(k_z^2\mathbf{u}_x\mathbf{u}_y - k_o^2\mathbf{u}_y\mathbf{u}_x) \tag{8}$$

satisfying

$$\overline{\overline{\mathsf{J}}}_t^2 = -\overline{\overline{\mathsf{I}}}_t, \quad \overline{\overline{\mathsf{I}}}_t = \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y. \tag{9}$$

The transverse components of the fields $\mathbf{E}^t, \mathbf{H}^t$ transmitted through the PEMC interface $z = 0$ satisfy the continuity conditions

$$\mathbf{E}_t^i + \mathbf{E}_t^r = \mathbf{E}_t^t = -M\mathbf{H}_t^t, \tag{10}$$

$$\overline{\overline{\mathsf{J}}}_t \cdot (\mathbf{E}_t^i - \mathbf{E}_t^r) = \eta_o\mathbf{H}_t^t, \tag{11}$$

from which we can eliminate $\mathbf{H}_t$ and obtain the reflected field as

$$\mathbf{E}_t^r = \overline{\overline{\mathsf{R}}}_t \cdot \mathbf{E}_t^i, \quad \overline{\overline{\mathsf{R}}}_t = \frac{1-(M\eta_o)^2}{1+(M\eta_o)^2}\overline{\overline{\mathsf{I}}}_t - \frac{2M\eta_o}{1+(M\eta_o)^2}\overline{\overline{\mathsf{J}}}_t. \tag{12}$$

The transverse component of the transmitted field at the interface becomes

$$\mathbf{E}_t^t = \frac{2}{1+(M\eta_o)^2}(\mathbf{E}_t^i - M\eta_o\overline{\overline{\mathsf{J}}}_t \cdot \mathbf{E}_t^i). \tag{13}$$

3. Fields in the PEMC medium

Starting from the exact PEMC conditions (1) does not yield unique transmitted fields, since the two Maxwell equations become the same equation. It is not much of an improvement to start from the conditions (3). In fact, eliminating $\mathbf{E}^t$ and $\mathbf{B}^t$ from the former of the Maxwell equations

$$\mathbf{k}^t \times \mathbf{E}^t = \omega\mathbf{B}^t, \qquad \mathbf{k}^t \times \mathbf{H}^t = -\omega\mathbf{D}^t, \tag{14}$$

yields $\mathbf{k}^t \times \mathbf{H}^t = -\omega(\mathbf{D}^t + \mathbf{H}^t/q)$, which together with the latter Maxwell equation leads to $\mathbf{H}^t/q = 0$. For finite q this would imply $\mathbf{H}^t = 0$ which eventually requires vanishing of all fields. To avoid that, we must assume $|q| = \infty$, whence (3) coincides with (1).

As a third alternative let us consider the Tellegen medium conditions (2), which inserted in the Maxwell equations as

$$\mathbf{k}^t \times \mathbf{H}^t = -\omega q(M\mathbf{E}^t + \mathbf{H}^t), \tag{15}$$

$$M\mathbf{k}^t \times \mathbf{E}^t = \omega q(M\mathbf{E}^t + \mathbf{H}^t), \tag{16}$$

yield

$$\mathbf{k}^t \cdot (M\mathbf{E}^t + \mathbf{H}^t) = 0, \quad \mathbf{k}^t \times (\mathbf{H}^t + M\mathbf{E}^t) = 0, \tag{17}$$

$$\mathbf{k}^t \times (\mathbf{k}^t \times \mathbf{E}^t) = 0, \quad \mathbf{k}^t \times (\mathbf{k}^t \times \mathbf{H}^t) = 0, \tag{18}$$

whence

$$\mathbf{k}^t \times (\mathbf{k}^t \times (M\mathbf{E}^t + \mathbf{H}^t)) = 0 \quad \Rightarrow \quad (\mathbf{k}^t \cdot \mathbf{k}^t)(M\mathbf{E}^t + \mathbf{H}^t) = 0. \tag{19}$$

Assuming $\mathbf{k}^t \cdot \mathbf{k}^t \neq 0$ yields $M\mathbf{E}^t + \mathbf{H}^t = 0$, $\mathbf{D}^t = \mathbf{B}^t = 0$ and $\mathbf{E}^t, \mathbf{H}^t$ parallel to $\mathbf{k}^t$. Such a special longitudinal wave cannot be excited by general incident fields, as can be shown by substituting in (13).

In the converse case the wave vector $\mathbf{k}^t$ is determined as

$$\mathbf{k}^t \cdot \mathbf{k}^t = 0, \quad \Rightarrow \quad \mathbf{k}^t = k_x(\mathbf{u}_x + j\mathbf{u}_z), \tag{20}$$

by requiring that the field does not grow exponentially for $z \to -\infty$. Thus, the transmitted field has the form of surface wave in the half space $z < 0$:

$$\mathbf{E}^t(\mathbf{r}) = \mathbf{E}^t \exp(k_x z) \exp(-jk_x x). \tag{21}$$

Expanding (18) we have $\mathbf{k}^t \cdot \mathbf{E}^t = 0$ and $\mathbf{k}^t \cdot \mathbf{H}^t = 0$ whence the axial electric fields are found as

$$E_z^t = jE_x^t, \qquad H_z^t = jH_x^t. \tag{22}$$

Substituting these in the transverse part of (16) we obtain the following relation between the transverse fields of the plane wave:

$$\mathbf{H}_t^t = -M\mathbf{E}_t^t + \frac{jk_x M}{\omega q}\mathbf{u}_x E_y^t. \tag{23}$$

At this point we can let $|q| \to \infty$, whence the fields satisfy the PEMC condition (1). Thus, (13) can be applied to define the complete transmitted electric field in terms of any given incident field as

$$\mathbf{E}^t = (\bar{\bar{\mathsf{I}}}_t + j\mathbf{u}_z\mathbf{u}_x) \cdot \mathbf{E}_t^t = \frac{2}{1 + (M\eta_o)^2}(\bar{\bar{\mathsf{I}}}_t + j\mathbf{u}_z\mathbf{u}_x) \cdot (\mathbf{E}_t^i - M\eta_o\bar{\bar{\mathsf{J}}}_t \cdot \mathbf{E}_t^i), \tag{24}$$

$$\mathbf{H}^t = -M\mathbf{E}^t. \tag{25}$$

The other transmitted field components are obtained from the Maxwell equations as

$$\mathbf{B}^t = \frac{1}{\omega}\mathbf{k}^t \times \mathbf{E}^t = \frac{2k_x(\mathbf{u}_z - j\mathbf{u}_x)}{\omega k_z(1 + (M\eta_o)^2)}(k_z E_y^i + M\eta_o k_o E_x^i), \tag{26}$$

$$\mathbf{D}^t = M\mathbf{B}^t = \frac{2Mk_x(\mathbf{u}_z - j\mathbf{u}_x)}{\omega k_z(1 + (M\eta_o)^2)}(k_z E_y^i + M\eta_o k_o E_x^i). \tag{27}$$

$\mathbf{B}^t$, and $\mathbf{D}^t$, are circularly polarized vectors.

Thus, by assuming the PEMC conditions as (2), the transmitted fields are uniquely obtained for any incident plane waves and they are expressed as physically plausible decaying waves which, however, carry along no energy because the Poynting vector has no real part for real M:

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{M}{2}\mathbf{E} \times \mathbf{E}^* = -\mathbf{S}^*. \tag{28}$$

4. Special cases

It is not often that fields inside the PEC or PMC media are considered and it may be erroneously assumed that the fields vanish in both cases. However, the PEC conditions do not require vanishing of $\mathbf{H}$ and $\mathbf{D}$ and the PMC conditions do not require vanishing of $\mathbf{E}$ and $\mathbf{B}$. Considering the fields excited by the incident plane wave, for the PMC half space we have $M = 0$, whence (24) – (27) become $\mathbf{H}^t = 0,\ \mathbf{D}^t = 0$ and

$$\mathbf{E}^t = 2(\bar{\bar{\mathsf{I}}}_t + j\mathbf{u}_z\mathbf{u}_x) \cdot \mathbf{E}_t^i, \qquad \mathbf{B}^t = \frac{2k_x}{\omega}(\mathbf{u}_z - j\mathbf{u}_x)E_y^i. \tag{29}$$

Similarly, for the PEC we obtain $\mathbf{E}^t = 0,\ \mathbf{B}^t = 0$ and

$$\mathbf{H}^t = \frac{2}{\eta_o}(\bar{\bar{\mathsf{I}}}_t + j\mathbf{u}_z\mathbf{u}_x)\cdot\bar{\bar{\mathsf{J}}}_t\cdot\mathbf{E}_t^i, \quad \mathbf{D}^t = \frac{2k_x k_o}{\omega k_z \eta_o}(\mathbf{u}_z - j\mathbf{u}_x)E_x^i. \tag{30}$$

Thus, for example, in the PEC half space, the incident plane wave excites a surface wave with the fields

$$\mathbf{H}^t(\mathbf{r}) = \frac{2}{\eta_o}(\bar{\bar{\mathsf{I}}}_t + j\mathbf{u}_z\mathbf{u}_x)\cdot\bar{\bar{\mathsf{J}}}_t\cdot\mathbf{E}_t^i\exp(k_x z)\exp(-jk_x x), \tag{31}$$

$$\mathbf{D}^t(\mathbf{r}) = \frac{2k_x k_o}{\omega k_z \eta_o}(\mathbf{u}_z - j\mathbf{u}_x)E_x^i\exp(k_x z)\exp(-jk_x x). \tag{32}$$

5. PEMC slab

The problem of the PEMC slab $0 > z > -d$ is an extension of the previous analysis. The plane-wave fields $\mathbf{E}^\tau, \mathbf{H}^\tau$ in the region $\zeta < -d$ satisfy at $z = -d$ the interface condition

$$\eta_o(M\mathbf{E}_t^\tau + \mathbf{H}_t^\tau) = (M\eta_o\bar{\bar{\mathsf{I}}}_t + \bar{\bar{\mathsf{J}}}_t)\cdot\mathbf{E}_t^\tau = \eta_o(M\mathbf{E}_t^t + \mathbf{H}_t^t) = 0, \tag{33}$$

whence $\mathbf{E}_t^\tau = 0$ because the dyadic in brackets has an inverse. This leads to vanishing of all field components in the half space $z < -d$. The fields in the slab now consist of two plane waves as

$$\mathbf{E}_t^t(\mathbf{r}) = \exp(-jk_x x)(\mathbf{E}_t^{t+}\exp(k_x z) + \mathbf{E}_t^{t-}\exp(-k_x z)), \tag{34}$$

$$\mathbf{H}_t^t(\mathbf{r}) = -M\exp(-jk_x x)(\mathbf{E}_t^{t+}\exp(k_x z) + \mathbf{E}_t^{t-}\exp(-k_x z)). \tag{35}$$

Because the transverse fields vanish at the interface $z = -d$, the z dependence in the PEMC slab must of the form

$$\mathbf{E}_t^t(\mathbf{r}) = \mathbf{E}_t^t\exp(-jk_x x)\sinh(k_x(z+d)). \tag{36}$$

The amplitude is obtained from the condition at the interface $z = 0$ as

$$\mathbf{E}_t^t = \frac{2(\mathbf{E}_t^i - M\eta_o\bar{\bar{\mathsf{J}}}_t\cdot\mathbf{E}_t^i)}{(1+(M\eta_o)^2)\sinh(k_x d)}. \tag{37}$$

6. Conclusions

The problem of finding fields inside the perfect electromagnetic conductor (PEMC) has encountered nonunique solutions when excited by exterior fields. In anticipation that it may be due to the definition of the medium conditions, in the present study a problem was attacked by considering three different definitions for the PEMC medium conditions, (1), (2) and (3). Considering an obliquely incident plane wave, it was shown that both (1) and (3) do not lead to a unique representation of fields while (2) appears satisfactory in this respect. In terms of (2) it was shown that the field in the PEMC half space has the form of a surface wave with exponentially decaying field normal to the interface. In the problem of PEMC slab, the interior field was shown to obey sinh dependence and the field transmitted through the slab is zero.

7. References

1. I.V. Lindell and A.H. Sihvola, "Perfect electromagnetic conductor," *J. Electromag. Waves Appl.*, vol.19, no.7, pp.861-869, 2005.

2. I.V. Lindell and A.H. Sihvola, "Transformation method for problems involving perfect electromagnetic conductor (PEMC) structures," *IEEE Trans. Antennas Propagat.*, vol.53, no.9, pp.3005–3011, September 2005.

3. F.W. Hehl and Yu.N. Obukhov, *Foundations of Classical Electrodynamics*, Boston: Birkhäuser, 2003.

4. B.D.F. Tellegen, "The gyrator, a new electric network element," *Philips Res. Repts.*, vol.3, no.2, pp.81–101, 1948.

5. B. Jancewicz, "Plane electromagnetic wave in PEMC", *J. Electro. Waves Appl.*, vol.20, no.5, pp.647–659, 2006.