Time domain Weyl's identity and transient modes due dipoles periodically distributed over a half space

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Abstract

Extensive research has been in conducted the analysis of the radiation by a periodic array over a half space in the frequency domain, but analogous results in time domain do not exist. In fact, obtaining expressions for the periodic Green's function directly in the time domain has only been a recent endeavor, and this has been done only for planar periodic distributions in free space. A traditional approach to obtaining this expression is to derive the corresponding Green's function in frequency domain and then transforming these to time. The approach that we espouse is slightly different. Our starting point will be the time domain Weyl's identity, which results in expressing the radiated field in terms of homogeneous and inhomogeneous plane waves. Application of the Floquet theorem and using a time symmetric source reduces this to a semi-infinite integral. The presence of the half-space is incorporated using reflection and transmission coefficients. In this paper, we intend investigating methods of possibly evaluating this integral and the physical interpretation thereof.

1. Introduction

Radiation by periodic structures find numerous applications ranging from periodic antennas to frequency selective surfaces to metamaterials. The analysis of these structures is well understood and there exists a plethora of papers that shed insight into the underlying physics [1, 2]. However, analyzing radiation/scattering due to these structures in time domain can provide interesting insight, in addition to being the basis for computationally efficient algorithms. However, the development of transient Floquet modes has only been a recent endeavor [3–5]. The starting point of these papers is the frequency domain Green's function represented in terms of Floquet modes and then developing different means of inverting this expression. All of these methods assume that a set of periodically excited dipoles reside in free space. More recently, the authors of this paper have developed an alternate method to solve this very problem, but starting with the time domain version of the Weyl's identity [6]. It was shown in Ref. [6] that it is possible to derive transient Floquet modes directly or using the so called causality trick. The latter is interesting as it results in a finite range integral. It was also shown that one could use the Whittaker integral [7] (which results in a superposition of the retarded and advanced potential) for simultaneously excited dipoles. As an aside, it is interesting to note that the use of time domain Weyl's identity to analyze fields radiated by time dependent source is not rampant in electromagnetics. More often, the spectral theory of transients and an analytic counterpart of the Whittaker integral are used [8, 9].

However, while [6] details the derivation of time domain Floquet modes using the Weyl's identity for dipoles residing in free space, its advantages or its use in deriving Floquet modes for dipoles over a layered medium has not been explored; an attempt that will be carried out here. At this point, it is not clear that different methods that have been presented in [6] can be easily extended to the layered medium problem. It is readily apparent that such a Green's function will find uses in most practical situations. This paper is organized as follows: In the next Section, we will introduce time domain Weyl's identity, then derive an expression for the periodic layered medium Green's function.

Finally, we will summarize this and future efforts in Section 3.

2. Formulation

Consider an infinite sequentially excited planar periodic dipole array that resides in the x–y plane. The inter-element spacings along the $\hat{x}$ and $\hat{y}$ direction will be denoted by d_x and d_y respectively. Without loss of generality, the origin is chosen to coincide with one of the dipoles. These phased array currents are denoted by $\mathbf{J}(\mathbf{r}, t) = \hat{J} J(\mathbf{r}, t)$ and expressed using notation similar to that in [5],

$$J(\mathbf{r}, t) = \sum_{m,n=-\infty}^{\infty} f(t) \star \delta\left(\mathbf{r}' - \mathbf{x}_{mn}\right) \delta\left(t - \mathbf{s} \cdot \mathbf{x}_{mn}/c\right) \tag{1}$$

where $\star$ denotes a temporal convolution, $f(t)$ describes the pulse shape and is of duration T, c is the speed of light in free space, $\mathbf{x}_{mn} = m d_x \hat{x} + n d_y \hat{y}$, and $\mathbf{s} = s_x \hat{x} + s_y \hat{y} = s\hat{s}$. The parameter $\mathbf{s}$ is related to the progression of delay (speed) of the switch-on time of the dipole array, and of interest is the case $s < 1$. In this paper, it is assumed that the dipoles are vertically oriented. Next, we will introduce fundamentals of analytic signals and the time domain Weyl identity.

2.1 Preliminaries

Given a Fourier transform pair $f(t)$ and $\mathcal{F}(\omega)$,

$$\begin{aligned} \mathcal{F}(\omega) &= \int_{-\infty}^{\infty} dt \exp(-j\omega t) f(t) \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \exp(j\omega t) \mathcal{F}(\omega) \end{aligned} \tag{2}$$

an analytic signal $F(\sigma)$ can be defined as,

$$F(\sigma) = \frac{1}{\pi} \int_{0}^{\infty} d\omega \exp(j\omega\sigma) \mathcal{F}(\omega), \quad \text{for } \Im\{\sigma\} > 0 \tag{3}$$

The real signal $f(t)$ and the analytic signal are related via

$$f(t) = \Re\{F(t)\}, \quad \text{for } t \in \tag{4}$$

As is well known, when $\mathcal{F}(\omega) = 1$ the corresponding real signal is the Dirac delta function $\delta(t)$. The associated analytic signal, or the so-called analytic delta function, is defined accordingly,

$$\Delta(\sigma) = \frac{1}{\pi} \int_{0}^{\infty} d\omega \exp(j\omega\sigma), \quad \text{for } \Im\{\sigma\} > 0 \tag{5}$$

The analytic delta function $\Delta(\sigma)$ plays a central role in the formulation.

Consider a time dependent source that radiates in free space. The potential at any point is given by the temporal and spatial convolution of the source with the retarded potential Green's function $\delta(t - R/c)/(4\pi R)$. As opposed to evaluating this convolution directly, one can alternatively derive the potential using the time domain Weyl identity, which can be written

$$\frac{\delta(t - R/c)}{4\pi R} = \Re\left\{\frac{\Delta(t - R/c)}{4\pi R}\right\} \quad \text{for } t > 0 \tag{6a}$$

$$\frac{\Delta(t - R/c)}{4\pi R} = \frac{-\partial_t}{8\pi^2} \iint_{-\infty}^{\infty} dp_x dp_y \frac{1}{p_z} \Delta\left[t - p_x(x - x') - p_y(y - y') - p_z|z - z'|\right] \tag{6b}$$

$$p_z = \sqrt{\frac{1}{c^2} - p_x^2 - p_y^2} = \begin{cases} |p_z| & \text{if } p_x^2 + p_y^2 \le 1/c^2 \\ -j|p_z| & \text{if } p_x^2 + p_y^2 > 1/c^2 \end{cases} \tag{6c}$$

where ∂_t denotes derivative with respect to time, and $R = |\mathbf{r} - \mathbf{r}'|$. This representation comprises of a superposition of both homogeneous and inhomogeneous time domain plane waves. As in the frequency domain, it follows that the above expression may be changed trivially by including the reflection/transmission coefficient to account for layered medium.

2.2 Dipole over a half space

For the problem analyzed herein, assume that the media in region $1(z > 0)$ and region $2(z < 0)$ are non-magnetic, and are characterized by relative dielectric constants ϵ_1 and ϵ_2, respectively, and the index of refraction is defined as $N = \sqrt{\epsilon_2/\epsilon_1}$. First, let's consider the radiation from a single vertical dipole located at point (x', y', z') in region 1. Using (6), and Floquet theorem, it follows that the Green's function for the reflected Hertz potential can be written as

$$\begin{aligned}
G^r_{mn}(t) &= A\int_0^\infty d\kappa \frac{R'(\kappa)}{jp'_{z1}(\kappa)} \exp\left[j(\kappa\tau - p'_{z1}(\kappa)\tau_0^r - \boldsymbol{\alpha}\cdot\boldsymbol{\rho})\right] \\
R'(\kappa) &= \frac{\chi p'_{z1}(\kappa) - p'_{z2}(\kappa)}{\chi p'_{z1}(\kappa) - p'_{z2}(\kappa)} \\
p'_{z1}(\kappa) &= \sqrt{(\kappa - \bar{\beta}_1)^2 - \beta_1^2} \\
p'_{z2}(\kappa) &= \sqrt{(\kappa - \bar{\beta}_2)^2 - \beta_2^2} \\
\chi &= \frac{N^2\sqrt{1-s^2}}{\sqrt{N^2 - s^2}}
\end{aligned} \tag{7}$$

for the m and nth mode. Using mode symmetry, and the analytic properties of the reflection coefficient, the real signal associated a superposition of these modes can be written as

$$\begin{aligned}
g^{r\pm}_{mn}(t) &= A\ \Re\left\{\int_0^\infty d\kappa R(\kappa)\frac{\exp\left[j(\kappa\tau - \sqrt{\kappa^2 - \beta_1^2}\tau_0^r)\right]}{j\sqrt{\kappa^2 - \beta_1^2}}\right\} \\
R(\kappa) &= \cos(\bar{\beta}_1\tau - \boldsymbol{\alpha}\cdot\boldsymbol{\rho})R_p + j\sin(\bar{\beta}_1\tau - \boldsymbol{\alpha}\cdot\boldsymbol{\rho})R_m
\end{aligned} \tag{8}$$

where the variables R_p and R_m are defined as

$$\begin{aligned}
R_p &= R'(\kappa + \bar{\beta}) + R''(\kappa - \bar{\beta}) \\
R_m &= R'(\kappa + \bar{\beta}) - R''(\kappa - \bar{\beta}) \\
R'(\kappa + \bar{\beta}) &= \frac{\chi\sqrt{\kappa^2 - \beta_1^2} - \sqrt{(\kappa + \bar{\beta}_1 - \bar{\beta}_2)^2 - \beta_2^2}}{\chi\sqrt{\kappa^2 - \beta_1^2} + \sqrt{(\kappa + \bar{\beta}_1 - \bar{\beta}_2)^2 - \beta_2^2}} \\
R''(\kappa - \bar{\beta}) &= \frac{\chi\sqrt{\kappa^2 - \beta_1^2} - \sqrt{(\kappa - \bar{\beta}_1 + \bar{\beta}_2)^2 - \beta_2^2}}{\chi\sqrt{\kappa^2 - \beta_1^2} + \sqrt{(\kappa - \bar{\beta}_1 + \bar{\beta}_2)^2 - \beta_2^2}}
\end{aligned} \tag{9}$$

The evaluation of this integral is far from trivial. The integral contains a branch point singularity. While analytical resolution of this integral may not be possible it can be evaluated numerically. An approximate closed form can be obtained by noticing that the R_p and R_m decay to a constant (in some physical settings they decay to zero) as κ approaches infinity. Details of the means to do so will be elaborated upon in the conference. While this is one method of solving this problem, it is not clear whether using causality trick would yield an integral that is simpler to evaluate. This question will also be answered in the conference.

3. Summary

This paper presents a method to derive time domain Floquet modes for a vertical dipole over a layered medium. Unlike traditional approaches, we start with the time domain Weyl's identity. An approach presented here is a straightforward application of this identity to this problem. However, this is not the only approach. One could, possibly, attempt using the causality trick. Details of the approach presented here as well as exploring other possible approaches to deriving these modes will be presented at the conference.

4. References

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