Compressive Sensing for Inverse Scattering

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1. Introduction

Compressive sensing is a new field in signal processing and applied mathematics. It allows one to simultaneously sample and compress signals which are known to have a sparse representation in a known basis or dictionary along with the subsequent recovery by linear programming (requiring polynomial (P) time) of the original signals with low or no error [1–3]. Compressive measurements or samples are non-adaptive, possibly random linear projections of the given signal. Most importantly, sparsity arises in many physical signals, hence this approach is of significant importance. The results in this area apply to biomedical imaging, astronomy, single-pixel photography, and many other disciplines.

The present work outlines certain aspects of our ongoing research in this area. Of much interest is the development of new approaches to linear and nonlinear inverse scattering problems [4] that are based on compressive sensing ideas. Past work in compressive sensing has been restricted to linear inverse problems of the form $y = Ax$ where A is a matrix mapping input (object) x to output (data) y. In this linear context, the focus has been to show that despite significant undersampling of the data signal y as projections of the form $y_0 = P^T y$, where T denotes the complex conjugate transpose and P is a measurement matrix obeying a mild incoherency property, one can for a broad class of sparse object signals x (where most of the entries of x are zero or negligible) still carry out perfect inversions or reconstructions with low error to the compressed inverse problem of inverting x from y_0 where $y_0 = P^T y = P^T Ax$. The focus of the present research is to study how these results apply to wave inverse scattering, in both the linear regime of the so-called Born approximation for weakly scattering objects as well as in the more general context of strongly scattering objects exhibiting non-negligible multiple scattering interactions. The derived developments are motivated mostly by detection and imaging applications (e.g., biomedical sensing, nondestructive testing, radar) requiring computationally non-intensive (P time) signal processing with limited wave data about a given event or target of interest.

Particular emphasis is given in the following to the framework termed Bayesian compressive sensing [1]. Inspired by this approach, next we derive new inverse scattering approaches based on maximum a posteriori probability (MAP) estimators for the unknown object function (the scattering potential representing the material constitutive properties) which is assumed to arise as a realization of a sparsity-inducing Laplace prior. In the resulting inversion methods, a 1-norm regularizing constraint substitutes the 2-norm constraint that is more typically adopted in inverse theory. This 1-norm constraint is typical of compressive sensing theory [1–3]. Two different frameworks are proposed: The first strategy corresponds to compressive sensing counterparts of conventional iterative Born methods. The second approach focuses on the more restricted goals of scatterer localization (for point targets) and shape reconstruction (for extended scatterers). This focus provides non-iterative approaches that are compressive sensing counterparts of the more familiar maximum likelihood, MUSIC, and other signal-subspace-based methods which have been an active research area of our group in recent years [5–8].

2. Bayesian Compressive Sensing for Inverse Scattering

The scattering matrix K for transmitters at points $\mathbf{R}_j^t, j = 1, 2, \cdots, N_t$ and receivers at $\mathbf{R}_i^r, i = 1, 2, \cdots, N_r$ is given by

$$K(i,j) = \int d\mathbf{x} G_0(\mathbf{R}_i^r, \mathbf{x}) V(\mathbf{x}) G(\mathbf{x}, \mathbf{R}_j^t) = \int d\mathbf{x} G(\mathbf{R}_i^r, \mathbf{x}) V(\mathbf{x}) G_0(\mathbf{x}, \mathbf{R}_j^t) \tag{1}$$

where G_0 is the Green function describing radiation and propagation in the background medium (without the unknown scatterers or inhomogeneities) and G is the total Green function (including the unknown medium contribution) which obeys $G(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}, \mathbf{x}') + \int d\mathbf{x}'' G_0(\mathbf{x}, \mathbf{x}'') V(\mathbf{x}'') G(\mathbf{x}'', \mathbf{x}')$. Clearly, the mapping from V to K in (1) is nonlinear. Henceforth K will be vectorized. Thus K will mean the $N_r N_t \times 1$ data vector $\mathrm{vec}(K)$.

Assuming that the sought-after scattering potential can be represented in a dictionary that is known a priori (e.g., the scatterers can be small, point-like targets in the computational grid, or the scattering potential can correspond to a biomedical or geophysical medium for which one has a priori knowledge of the dictionary in question), then we write: $V(\mathbf{x}) = \sum_{l=1}^{L} \tau(l)\phi_l(\mathbf{x})$. For example, for point targets $\phi_l(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{X}_l)$ where $\mathbf{X}_l$ represents the lth target position. It is assumed that τ is sparse, i.e., most of its entries are zero. But which entries are zero is not known a priori. Introducing

$$\Phi_{l,\tau}(i,j) = \int d\mathbf{x} G_0(\mathbf{R}_i^r, \mathbf{x})\phi_l(\mathbf{x})G(\mathbf{x}, \mathbf{R}_j^t) \tag{2}$$

then (1) yields $K(i,j) = \sum_l \tau(l)\Phi_{l,\tau}(i,j)$ which gives $\mathrm{vec}(K) = \sum_l \tau(l)\mathrm{vec}(\Phi_{l,\tau})$. Henceforth $\mathrm{vec}(\Phi_{l,\tau})$ will be denoted simply as $\Phi_{l,\tau}$ with the understanding that this is the vectorized form of this matrix. It is convenient for notational compactness to re-write the preceding results as $K = A_\tau \tau$ where A_τ is the $N_r N_t \times L$ matrix $A_{l,\tau} = [\Phi_{1,\tau}\ \Phi_{2,\tau}\ ...\ \Phi_{L,\tau}]$. Note that if the scattering is weak such that it can be approximated via the Born approximation, then $G \simeq G_0$ so that from (2) the quantity $\Phi_{l,\tau}$ is independent of τ and depends only of l, i.e., then we can put $\Phi_{l,\tau} = \Phi_l$. This renders the special linear model $K = A\tau$ where $A = [\Phi_1\ \Phi_2\ ...\ \Phi_L]$. We are also interested in this particular weak scattering regime, but our focus is the more general regime including multiple scattering where the signal model is nonlinear.

The compressed data are assumed to take the form $g = P^T K$ where T denotes the complex conjugate transpose and P is a $N_r N_t \times v$ matrix whose v columns define v projective measurements of the $N_r N_t \times 1$ data vector K. Of particular interest are random projective measurements which can be suitably converted to random orthoprojectors. Adding complex white Gaussian noise w of variance σ^2 then we have the revised model $g = P^T K + P^T w = P^T A_\tau \tau + P^T w$. Our goal is to derive MAP estimators for τ that take into account the a priori knowledge on sparsity (most of the entries of τ are zero). To this end, consider the Laplace probability density (sparsity prior)

$$p(\tau|\lambda) = (\lambda/2)^L \exp(-\lambda \sum_l |\tau(l)|) = (\lambda/2)^L \exp(-\lambda ||\tau||_1) \tag{3}$$

where λ is a (regularizing) parameter which characterizes the sparsity level. Thus for large values of λ the density in (3) favors low 1-norm signals, which in many instances also carries low 0- norm (this is, in fact, the key useful fact in many of the most impressive developments in compressive sensing [2, 3]).

The MAP estimator $\hat{\tau}$ for τ is then given for orthogonal projections P by

$$\hat{\tau} = \arg\min_\tau \left[\frac{v}{2N_r N_t \sigma^2} ||g - P^T A_\tau \tau||_2^2 + \lambda ||\tau||_1 \right]. \tag{4}$$

Note the role of the 1-norm regularization. Expression (4) contrasts with the more conventional solution obtained under the assumption of a normal or Gaussian prior, which is of the same form but where 1-norm of τ is substituted by 2-norm. The result (4) invites for some new iterative methods involving 1-norm regularization. Thus we can adopt an iterative scheme of the form:

$$\tau_{i+1} = \arg\min_\tau \left[\frac{v}{2N_r N_t \sigma^2} ||g - P^T A_{\tau_i} \tau||_2^2 + \lambda ||\tau||_1 \right] \tag{5}$$

where A_τ is computed at the $(i+1)$th iteration using the estimate for τ from the preceding iteration (τ_i). This value τ_i is used to obtain an estimate of the total Green function G from which A_{τ_i} is computed. An alternative approach is

$$\tau_{i+1} = \tau_i + \arg\min_{\Delta\tau} \left[\frac{v}{2N_r N_t \sigma^2} ||g - P^T A'_{\tau_i} \Delta\tau||_2^2 + \lambda ||\Delta\tau||_1 \right] \tag{6}$$

where $A'_\tau = [\Phi'_{1,\tau}\ \Phi'_{2,\tau}\ ...\ \Phi'_{L,\tau}]$ where

$$\Phi'_{l,\tau} = \int d\mathbf{x} G(\mathbf{R}_i^r, \mathbf{x})\phi_l(\mathbf{x})G(\mathbf{x}, \mathbf{R}_j^t). \tag{7}$$

Thus A'_{τ_i} is defined as above, but using τ_i as our ith approximation to τ from which we compute for the iteration in question the total Green function G which plays the role of our new background Green function. The methods (5,6)

are Bayesian compressive sensing versions of the conventional iterative Born methods, where we incorporate both the linear projective measurements, with the idea of reducing the number of measurements, as well as the sparsity prior which gives 1-norm regularization in place of the more familiar 2-norm regularization.

We develop next another scheme that is non-iterative. This approach belongs to the general class of so-called qualitative methods in inverse scattering [4], and has been studied before in [9] in the passive sensing regime of the inverse source problem. The following developments apply to the active sensing case relevant to the inverse scattering problem. The method has been designed for scatterers whose response can be modeled by a few dominant scattering centers. To ease exposition, we consider next the canonical case of scatterers formed by collections of point targets. In this particular case, one can borrow from results derived in [6] to show that the signal model in (1) can be expressed as

$$K = \Gamma_0 \mathrm{vec}(B) \tag{8}$$

where B is an $L \times L$ two-point scattering operator having entries

$$B(l, l') = \tau(l)\delta_{l,l'} + \tau(l)\tau(l')G(\mathbf{X}_l, \mathbf{X}_{l'}) \tag{9}$$

and Γ_0 is an $N_r N_t \times L^2$ matrix whose columns are the $N_r N_t \times 1$ vectors formed by vectorizing the matrices $\Psi_{l,l'}(i,j) = G_0(\mathbf{R}_i^r, \mathbf{X}_l) G_0(\mathbf{X}_{l'}, \mathbf{R}_j^t), l = 1, \cdots, L; l' = 1, \cdots, L$. A MAP-like estimator which ignores the model-based structure of B in (9) and assumes Laplace pdf-induced sparsity of B (a condition that requires a few point targets relative to the sensing array size) is then defined, for the noise signal model adopted earlier (i.e., $g = P^T K + P^T w$), by

$$\hat{B} = \arg\min_B \left[\frac{v}{2N_r N_t \sigma^2} ||g - P^T \Gamma_0 \mathrm{vec}(B)||_2^2 + \lambda ||\mathrm{vec}(B)||_1 \right] \tag{10}$$

which suggests the imaging function $P(\mathbf{X}_l) = \max_{l'} |\hat{B}(l, l')|$. This method has a simple structure under the Born approximation where it reduces to the more familiar Bayesian compressive sensing approach to inverting linear mappings.

The latter method was validated with computational examples based on the geometry shown in Figure 1. Figure 2 shows the imaging via (10) when all the 11 transmitters were used in generating the data and noise was added to achieve a signal-to-noise ratio of 18 dB (where the signal-to-noise ratio was defined as in [5]). Figure 3 shows a further compressed data set where only 1 transmit experiment (only the first transmit antenna from left to right in Figure 1 was active) was considered (under noiseless conditions) and a corresponding inverse source method of an earlier paper [9] was used. In running these examples we implemented the algorithms above in the familiar basis pursuit format, but we are currently working on more robust implementations based on suitable regularizations along the lines of L curve and other methods as implemented in [5]. We are also investigating learning machine approaches for implementation of the algorithms developed in this work.

3. Acknowledgment

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4. References

[1] S. Ji, Y. Xue, and L. Carin, “Bayesian compressive sensing”, IEEE Trans. Signal Processing, in press.

[2] E.J. Candès, J. Romberg, T. Tao, “Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information”, IEEE Trans. Inform. Theory, Vol. 52, pp.489-509, 2006.

[3] D.L. Donoho, “Compressed sensing”, IEEE Trans. Inform. Theory, Vol. 52, pp.1289-1306, 2006.

[4] F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer, Berlin, 2006.

[5] E.A. Marengo, F.K. Gruber, and F. Simonetti, “Time reversal MUSIC imaging of extended targets”, *IEEE Transactions on Image Processing*, Vol. 16, p. 1967-1984, 2007.

[6] E.A. Marengo and F.K. Gruber, “Subspace-based localization and inverse scattering of multiply scattering point targets”, *EURASIP J. Advances in Signal Processing*, Vol. 2007, Article ID 17342, 16 pages, 2007.

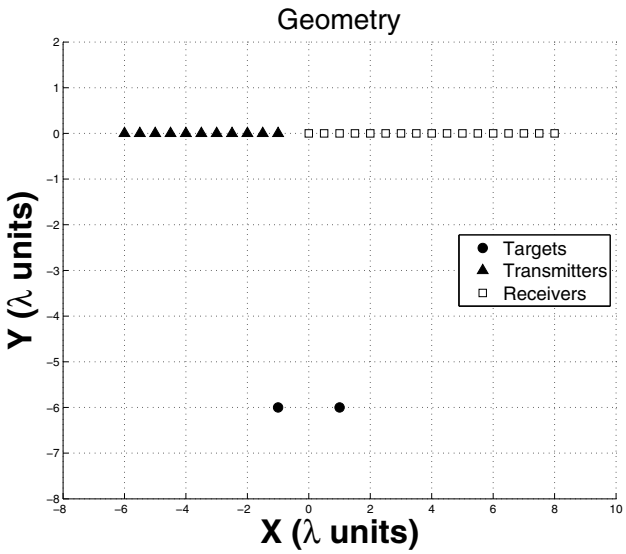

Figure 1. Geometry in two dimensional free space.

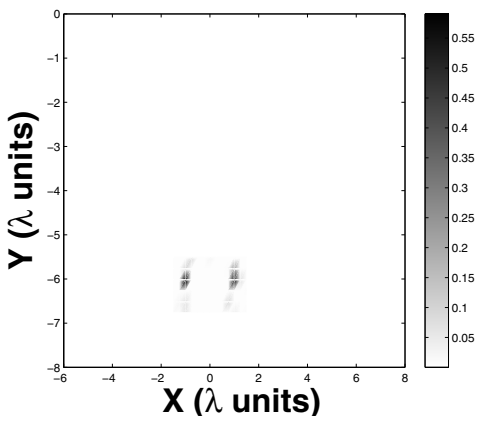

Figure 2. Imaging via (10) when all the 11 transmitters were used in generating the data.

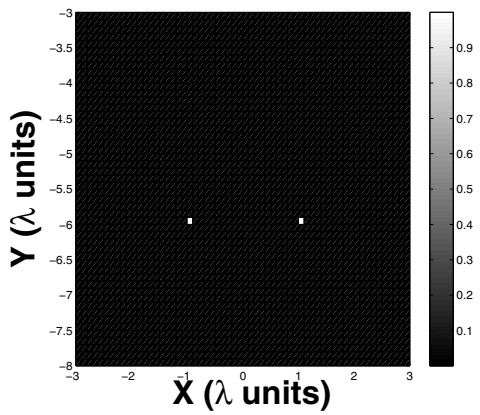

Figure 3. Reconstruction of τ using only 1 transmitter.

[7] E.A. Marengo, "Further theoretical considerations for time-reversal MUSIC imaging of extended scatterers", invited talk at the Special Session on Time Reversal of the *IEEE Statistical Signal Processing Workshop 2007*, Madison, Wisconsin, August 26-29, 2007 (preprint available at the website: www.ece.neu.edu/∼emarengo).

[8] E.A. Marengo and F.K. Gruber, "Non-iterative analytical formula for inverse scattering of multiply scattering point targets", *J. Acoust. Soc. Am.*, Vol. 120, p. 3782-3788, 2006.

[9] E.A. Marengo, "Inverse scattering by compressive sensing and signal subspace methods", Invited talk at the Special Session on "Wave-based Signal Processing", IEEE Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP) 2007, St. Thomas, U.S. Virgin Islands, Dec. 12-14, 2007 (preprint available at the website: www.ece.neu.edu/∼emarengo).