Acceleration of Direct Integral Equation Based Solvers for Scattering Problems

Amir Boag

School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel
Tel: +972-3-6408246, Fax: +972-3-6423508, boag@eng.tau.ac.il

Abstract

Direct solution of the linear systems of equations stemming from the method of moments (MoM) discretization of the integral equations of electromagnetic scattering is accelerated by a three step solution process. First, the non-uniform grid (NG) approach and rank-revealing orthogonal matrix decomposition are used in tandem to compress the MoM matrix by effectively removing the oversampling of conventional low-order discretization schemes and adaptively filtering out the non-interacting degrees of freedom. Then, the compressed system of equations is solved directly for the interacting currents and, finally, the non-interacting local currents are determined.

1. Introduction

Problems of electromagnetic scattering are often formulated using boundary or volumetric integral equations, which are subsequently converted to systems of linear equations using the method of moments (MoM). MoM discretization of the integral equations of computational electromagnetics conventionally requires roughly ten unknowns per linear wavelength. Even higher density of discretization is needed in the quasi-static regime or for non-smooth geometries. On the other hand, only two unknowns per wavelength are sufficient to describe the number of degrees of freedom of the scattered or radiated field. The contrast between the local sampling requirements and the radiated field properties that affect the interaction between spatially separated domains can be exploited to reduce the complexity of solving the MoM equations. An efficient compression technique based on the construction of radiating and non-radiating basis and testing functions has been proposed in [1]. In this work, we propose to construct locally and globally interacting basis and testing functions (similar to those in [2]) via a numerically efficient algorithm based on the non-uniform grid (NG) sampling and interpolation of radiated fields [3].

2. Formulation

Consider a boundary integral equation describing time-harmonic scattering by an arbitrary shaped body. MoM discretization of the integral equation leads to a system of linear equations $Zi = v$, where Z is the generalized impedance matrix, v is the excitation vector related to the incident field on the scatterer surface, and i is the vector of unknown expansion coefficients. As a first step, the scatterer boundary S is decomposed into subdomains of roughly equal size $S = \bigcup_{p=1}^{P} \overline{S}_p$. We assume that the MoM uses localized basis and testing functions, each of which can be assigned by an appropriate rule to a specific subdomain. For a subdomain p the matrix can be partitioned as shown in Fig. 1, where diagonal block Z_{pp} represents the local interactions within the subdomain p while the off-diagonal blocks $Z_{\hat{p}p}$ and $Z_{p\hat{p}}$ describe the interaction of the subdomain with the rest of the scatterer $\hat{S}_p = S \setminus \overline{S}_p$. The rank of these off-diagonal blocks is expected to be much smaller than their dimensions because of the oversampling inherent to the conventional discretization schemes involving low-order basis and testing functions. Note that subdomain *p* is shown in the figures as the first subdomain only for the sake of convenience.

In order to efficiently analyze the off-diagonal blocks without actually computing their elements, an NG is constructed for each subdomain. An example of NG for a subdomain of a two-dimensional problem of scattering by a circular arc is shown in Fig. 2. The NG allows for the computation of the field (produced by the currents confined to that subdomain) at any point of the scatterer by interpolation. The NG points allow

interpolation in the far-zone points of the scatterer while the near-zone points are computed directly. We can, therefore, express the off-diagonal block describing the coupling of subdomain p to the rest of the scatterer as

$$Z_{\hat{p}p} \approx T_p G_p : \bar{S}_p \to \mathcal{G}_p \to \hat{S}_p \tag{1}$$

where G_p is the matrix describing the field computation on $\mathcal{G}_p$, which comprises the NG points and the near field testing functions pertinent to $\bar{S}_p$. Also in (1), T_p is the testing and interpolation operator for the NG points. Note that T_p reduces to identity for the near field testing functions. In fact, the existence of T_p is sufficient for our purposes as we will not explicitly employ it in the formulation.

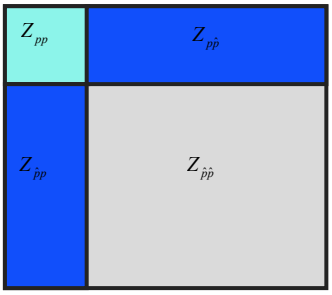

Figure 1: MoM Matrix partitioning for subdomain $\bar{S}_p$ and the rest of the scatterer.

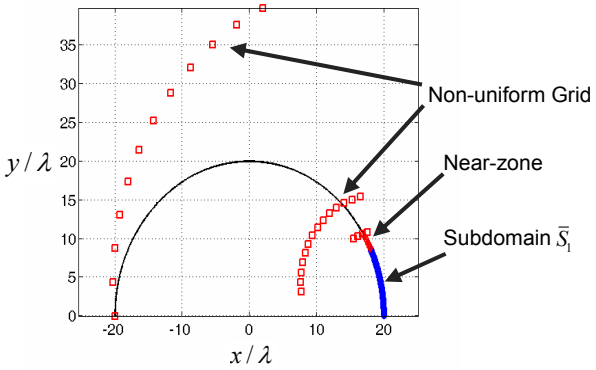

Figure 2: Polar non uniform grid (NG) and near-zone points (red) built around a subdomain (blue).

The analysis of the field computation on the NG can be used to determine the actual degrees of freedom involved in the interaction between the subdomain and the rest of the scatterer. This entails that the local and global interacting basis functions can be constructed based on a rank revealing factorization of the matrix describing field computation at the NG points and near field testing functions. Singular value decomposition (SVD) or a more efficient rank-revealing QR decomposition can be employed in this analysis. Using the SVD we obtain $G_p = U_p \Sigma_p V_p^{\dagger}$, where U_p and V_p are unitary matrices, Σ_p is a diagonal matrix containing the singular values σ_i arranged in the descending order, and dagger denotes the Hermitian conjugate. A given threshold τ determines the numerical rank r_p such that $\sigma_{r_p}/\sigma_1 > \tau$ and $\sigma_{r_p+1}/\sigma_1 < \tau$. Thus, we can suppress the small singular values beyond r_p by replacing Σ_p with $\Sigma_p^{(r_p)}$ where the small singular values σ_i, $i > r_p$ are replaced with zeros. This means that left multiplication by V_p provides the basis transformation $Z_{\hat{p}p} V_p = T_p G_p V_p \approx T_p U_p \Sigma_p^{(r_p)}$ that introduces column compression of the off-diagonal block $Z_{\hat{p}p}$. Formally, the column compression by basis transformation for all subdomains can be expressed as a right-multiplication of Z by a block diagonal matrix V involving all V_p, $p = 1, \ldots, P$. Obviously, the zero blocks in V are not used in the actual implementation of the matrix compression. Employing a reciprocal argument the off diagonal block $Z_{p\hat{p}}$ can be row compressed by a similar testing function transformation for subdomain p. The NG used in basis transformation can be used to place elemental dipole sources, whose fields over $\bar{S}_p$ can represent all fields of far-zone sources. The near-zone basis functions are used directly to complement the NG sources and create over $\bar{S}_p$ a complete set of fields representing those of all basis functions located outside this subdomain. Assuming that G_p^T is the matrix describing the fields over $\bar{S}_p$ due to the NG sources and near-zone basis functions, the desired row compression of $Z_{p\hat{p}}$ can be achieved by a left-multiplication by V_p^T, i.e., $V_p^T Z_{p\hat{p}} = V_p^T V_p^* \Sigma_p^{(r_p^T)} U_p^T T_p^T \approx \Sigma_p^{(r)} U_p^T T_p^T$ where V_p^T, U_p^T, and Σ_p^T are obtained by the SVD of the transpose of G_p^T. Combining matrices V_p^T for all subdomains, we

construct a block diagonal matrix V^T providing the desired row compression for the whole MoM matrix. The column and row compression can be expressed now as

$$A = V^T Z V, \qquad x = V^\dagger i, \qquad b = V^T v \tag{2}$$

where matrix A has zero rows and columns as depicted for subdomain p in Fig. 3. Also in (2), b stands for the transformed excitation vector and x is the new vector of unknowns. For symmetric MoM matrices obtained by using, for example, the electric field integral equation with Galerkin choice of equal basis and testing functions, the row compression of $Z_{p\hat{p}}$ does not require any additional analysis. The same matrices constructed for the basis transformation can be used for the testing function transformation. In this case, superscript T in the above expressions stands for simple matrix transpose.

The basis and testing function transformations produce a new diagonal block $A^{pp} = V_p^T Z_{pp} V_p$ and the corresponding excitation and unknowns vectors $b^p = V_p^T v_p$ and $x^p = V_p^\dagger i_p$. If r_p and r_p^T are not equal, let r_p represent the maximum of the two. The unknowns x^p, the excitations vector b^p and matrix block A^{pp} associated with the subdomain can be split into two groups. A group of r_p unknowns and excitations that interact with the rest of the scatterer and the remaining local ones:

$$A^{pp} = \begin{pmatrix} A_{11}^{pp} & A_{12}^{pp} \\ A_{21}^{pp} & A_{22}^{pp} \end{pmatrix}, \qquad x^p = \begin{pmatrix} x_1^p \\ x_2^p \end{pmatrix}, \qquad b^p = \begin{pmatrix} b_1^p \\ b_2^p \end{pmatrix} \tag{3}$$

The new off-diagonal block $A^{p\hat{p}}$ has only r_p non-zero rows designated $A_1^{p\hat{p}}$. Thus, the unknowns corresponding to the local basis functions are expressed in terms of the globally interacting ones

$$x_2^p = (A_{22}^{pp})^{-1} b_2^p - (A_{22}^{pp})^{-1} A_{21}^{pp} x_1^p \tag{4}$$

and can be effectively eliminated from the system of equations as illustrated in Fig. 4.

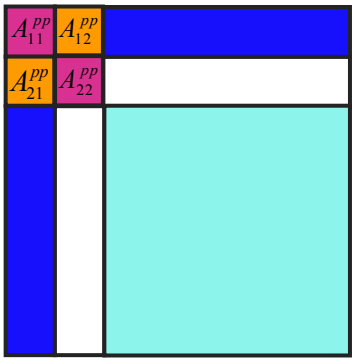

Figure 3: MoM Matrix with compressed off-diagonal blocks.

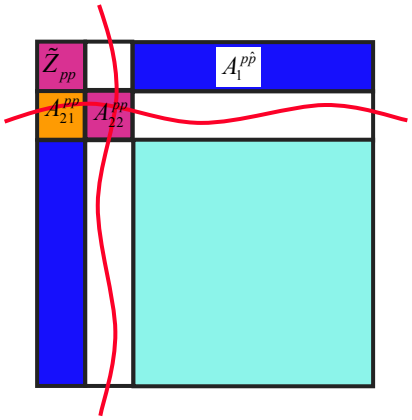

Figure 4: Schur's complement elimination of local unknowns and testing functions in the MoM Matrix.

The new diagonal block known as the Schur's complement and the new excitation vector are given by

$$\tilde{Z}_{pp} = A_{11}^{pp} - A_{12}^{pp} (A_{22}^{pp})^{-1} A_{21}^{pp}, \qquad \tilde{v}_p = b_1^p - A_{12}^{pp} (A_{22}^{pp})^{-1} b_2^p. \tag{5}$$

The compressed matrix comprises diagonal blocks $\tilde{Z}_{pp}$ and off-diagonal blocks $\tilde{Z}_{p\hat{p}} = A_1^{p\hat{p}}$ for $p = 1,\ldots,P$. The new excitation vector is constructed by concatenating $\tilde{v}_p$ for all subdomains and the unknowns vector involves all the global unknowns $\tilde{i}_p = x_1^p$. Application of the above procedure produces a much smaller system of equations, which for moderately sized problems can be solved directly. Once the interacting currents have been found the local currents can be determined via (4), thus, completing the solutions process.

3. Numerical Results

The matrix compression technique outlined in the previous section has been applied to a problem of two-dimensional scattering by a perfectly conducting semi circular arc 20λ in radius (shown in Fig. 2) illuminated by a transverse magnetic (TM) polarized field. The scatterer discretized with $N = 700$ unknowns has been subdivided into 7 subdomains each containing 100 unknowns. On the other hand, arc length of each subdomain is about 9λ and thus roughly 18 unknowns are needed to describe the interactions of the subdomain with the rest of the scatterer. Indeed with a relative tolerance of $\tau = 10^{-3}$, this is approximately the number of interacting unknowns retained by the matrix compression process. The relative error in the computed current is approximately 4×10^{-3} and the scattering cross section is even more accurate. Our study has shown that tightening the tolerance improves the relative accuracy while slightly increasing the size of the compressed matrix. The results are not sensitive to the specific choice of the parameters defining the NG. Only when very high accuracy is required an oversampled NG is needed.

4. Conclusion

For arbitrary geometries, the asymptotic computational complexity of the direct solver using the NG-based compression remains of $O(N^3)$, where N is the number of unknowns, however, the constant multiplier implicit in such complexity estimates is greatly reduced. In fact, thanks to the proposed matrix compression, the complexity multiplier is much smaller than unity. In addition, the proposed technique is geometrically adaptive and very high compression and truly fast direct solvers are possible for geometries of reduced dimensionalities such as elongated and quasi-planar configurations.

5. Acknowledgments

This research is sponsored in part by NATO's Public Diplomacy Division in the framework of “Science for Peace" programme and Israel Science Foundation under Grant 745/07.

6. References

1. F. X. Canning and K. Rogovin, “A universal matrix solver for integral-equation-based problems,” *IEEE Antennas Propagat. Mag.,* **45(1)**, Feb. 2003, pp. 19-26.
2. A. Zhu, R. J. Adams, and F. X. Canning, “Schur factorization of the impedance matrix in a localizing basis”, *J. of Electromagn. Waves and Appl*., **20(3)**, Jul. 2006, pp. 351-362.
3. A. Boag, E. Michielssen, and A. Brandt, “Nonuniform polar grid algorithm for fast field evaluation“, *IEEE Antennas Propagat. Lett*., **1**, Oct. 2002, pp. 142-145.