Some paradoxes associated with a recent summation rule in scattering theory

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Abstract

This paper reports on some peculiarities associated with a recently published summation rule for scattering of electromagnetic waves. The summation rule states that the extinction cross section integrated over all frequencies is equal to the low-frequency response of the target. Although the summation rule is intriguing by itself, it becomes even more paradoxical when a static conductivity model or the PEC boundary condition is introduced. The paradoxical character lies in the fact that the extinction cross section integrated over all frequencies is independent of the static conductivity as long as it is non-zero. This puzzling result is explained by rejecting the static conductivity model at zero frequency as suggested by numerical simulations of a homogeneous and isotropic sphere. In addition, the low-frequency behavior of diamagnetic materials is investigated using Herglotz functions and arguments from the theory of special relativity.

1 Introduction

Under the assumption of linearity, passivity, and time-translational invariance, a summation rule for scattering of electromagnetic waves is derived in Refs. 1 and 2 from the holomorphic properties of the forward scattering dyadic. The result states that the extinction cross section (*i.e.*, the sum of the scattering and absorption cross sections) integrated over all frequencies is equal to the static polarizability dyadics of the target. As a consequence, for a given target, there is only a limited amount of scattering and absorption available in any frequency interval. This far-reaching conclusion is applicable to a broad range of problems in theoretical physics involving wave interaction with matter. The summation rule also holds with minor changes to a large class of causal and reciprocal antennas [3]. Compared with the classical antenna bounds, the theory set forth in Ref. 3 yields sharper inequalities, and, more importantly, a new fundamental understanding of antenna dynamics solely based on its low-frequency expansion.

Consider a homogenous and isotropic sphere of radius a, and let $\kappa = ka$, where k denotes the angular wave number in free space. Introduce $\nu(\kappa) = \sigma_{\mathrm{ext}}(\kappa)/\pi a^2$ as the extinction cross section normalized with the geometrical cross section area πa^2. Let $\epsilon = \epsilon(\kappa)$ and $\mu = \mu(\kappa)$ measure the permittivity and permeability of the target relative to free space, and assume ϵ and μ are continuous at $\kappa = 0$. The summation rule in Ref. 1 then reads

$$\int_0^\infty \frac{\nu(\kappa)}{\kappa^2}\,\mathrm{d}\kappa = 2\pi\left(\frac{\epsilon(0)-1}{\epsilon(0)+2} + \frac{\mu(0)-1}{\mu(0)+2}\right). \tag{1}$$

Although the integral in (1) has a simple closed-form expression, this is generally not true for the integrand $\nu(\kappa)/\kappa^2$ for a fixed κ. In particular, note that the right hand side of (1) is independent of any temporal dispersion (although the integrand is not), depending only on the low-frequency response of the target. As expected, the right hand side of (1) vanishes for $\epsilon(0) = \mu(0) = 1$ which means that the target reduces to free space, *cf.*, the Kramers-Kronig relations in Ref. 4. More generally, if the equality in (1) is replaced by a less than or equal to, the new inequality holds also for any isotropic and homogeneous scatterer circumscribed by a sphere of radius a [1].

Under the assumption of continuity in the low-frequency limit, it follows from the Kramers-Kronig relations that $\epsilon(0)$ and $\mu(0)$ are bounded from below by the instantaneous responses $\epsilon_\infty = \lim_{\kappa\to\infty}\epsilon(\kappa) \geq 1$ and $\mu_\infty = \lim_{\kappa\to\infty}\mu(\kappa) \geq 1$, respectively. Since ϵ_∞ and μ_∞ are non-unique from a modeling point of view, see Ref. 5, it is sufficient to put $\epsilon_\infty = \mu_\infty = 1$. Thus, $\epsilon(0)$ and $\mu(0)$ are bounded from below by unity, and it follows that the right hand side of (1) is non-negative . This conclusion is consistent with the fact that the integrand $\nu(\kappa)/\kappa^2$ by definition is non-negative [1, 4].

However, the summation rule (1) is not valid if either ϵ or μ are discontinuous at $\kappa = 0$ which is the case for the static conductivity model [6, pp. 14–19]. Numerical results in Ref. 2 with temporal dispersive material parameters in the form of a Drude model, *i.e.*, a difference between a static conductivity model and a Debey model, suggest that the left hand side of (1) is independent of the static conductivity as long as it is non-zero. Furthermore, diamagnetic materials have a low-frequency permeability less than unity which seem to contradict (1) since the second term on the right hand side then becomes negative. The objective of this paper is to clarify the effects of static conductivity (including the PEC boundary condition) and diamagnetic material parameters in the context of (1) and the theory set forth in Refs. 1 and 2.

2 The effects of static conductivity in the low-frequency limit

Introduce the dimensionless quantity $\varsigma = \sigma a \eta > 0$, where σ denotes the static conductivity and η is the wave impedance in free space. Let $\epsilon' = \epsilon'(\kappa)$ be an arbitrary complex-valued permittivity such that ϵ' is continuous at $\kappa = 0$. Consider the target introduced in Sec. 1 with the following permittivity model which is singular at $\kappa = 0$:

$$\epsilon(\kappa) = \epsilon'(\kappa) + \mathrm{i}\frac{\varsigma}{\kappa}. \tag{2}$$

Without loss of generality, let the target be non-magnetic in the sense that $\mu = 1$ independent of κ.[1] Then it follows that the transition matrix is diagonal with electrical 2^ℓ-pole (dipole, quadrupole, ...) elements given by [7]

$$t_{2\ell}(\kappa) = -\frac{\mathrm{j}_\ell(\kappa)(\kappa\sqrt{\epsilon(\kappa)}\mathrm{j}_\ell(\kappa\sqrt{\epsilon(\kappa)}))' - \epsilon(\kappa)(\kappa \mathrm{j}_\ell(\kappa))'\mathrm{j}_\ell(\kappa\sqrt{\epsilon(\kappa)})}{\mathrm{h}_\ell^{(1)}(\kappa)(\kappa\sqrt{\epsilon(\kappa)}\mathrm{j}_\ell(\kappa\sqrt{\epsilon(\kappa)}))' - \epsilon(\kappa)(\kappa \mathrm{h}_\ell^{(1)}(\kappa))'\mathrm{j}_\ell(\kappa\sqrt{\epsilon(\kappa)})}, \tag{3}$$

where $\ell = 1, 2, \ldots$, and a prime denotes differentiation with respect to either $\kappa\sqrt{\epsilon(\kappa)}$ or κ depending on the arguments of j_ℓ and $\mathrm{h}_\ell^{(1)}$.[2] Here, j_ℓ and $\mathrm{h}_\ell^{(1)}$ denote the spherical Bessel and Hankel functions of first kind, respectively [8, §10]. The normalized extinction cross section is related to the transition matrix via $\nu(\kappa) = -2\,\mathrm{Re}\sum_\ell(2\ell+1)(t_{1\ell}(\kappa) + t_{2\ell}(\kappa))/\kappa^2$, *cf.*, the classical Mie series solution.

For the homogeneous and isotropic sphere, the right hand side of (1) is equal to $\gamma/2$, where γ denotes the degenerate eigenvalues of the high-contrast polarizability dyadic [1, 2]. This quantity is defined by the following low-frequency limit for the lowest order ($\ell = 1$) transition matrix element:

$$\gamma = -6\pi\mathrm{i}\lim_{\kappa\to 0}\frac{t_{21}(\kappa)}{\kappa^3}. \tag{4}$$

For a permittivity model which is continuous at $\kappa = 0$, it is straightforward to prove that $\gamma/2$ reduces to the right hand side of (1). The corresponding limit for the permittivity model (2) is somewhat more complicated as the asymptotic expansion $\sqrt{\epsilon(\kappa)} = \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\varsigma/\kappa} + \mathcal{O}(\sqrt{\kappa})$ when $\kappa \to 0$ must be inserted into (3). Together with the asymptotics $\mathrm{j}_\ell(\kappa) = 2^\ell \ell!\kappa^\ell/(2\ell+1)! + \mathcal{O}(\kappa^{\ell+2})$ and $\mathrm{h}_\ell^{(1)}(\kappa) = -\mathrm{i}(2\ell)!/2^\ell\ell!\kappa^{\ell+1} + \mathcal{O}(\kappa^{-\ell+1})$ as $\kappa \to 0$, see p. 437 in Ref. 8, it is not hard to show that $\gamma = 4\pi$. Thus, as that the left hand side of (1) is equal to $\gamma/2$, it is concluded that

$$\int_0^\infty \frac{\nu(\kappa)}{\kappa^2}\,\mathrm{d}\kappa = 2\pi. \tag{5}$$

So, as long as an arbitrarily small static conductivity $\varsigma > 0$ is present in the target, the integral on the left hand side of (1) is equal to 2π independent of the choice of $\epsilon'(0)$ and ς. Here, the paradoxical character lies in the fact that the integral on the left hand side of (1) is discontinuous in the limit as $\varsigma \to 0$. This is a severe restriction in the sense that there is no longer a freedom to model an electrical insulator as having a low value of ς or no static conductivity at all.

The transition matrix method described above is used to verify (5) by computing the extinction cross section. The result is depicted on the left hand side of Fig. 1 (the right figure is a close-up of the left figure for low frequencies) for $\epsilon' = 1$ and $\varsigma \in \{0.1, 1, 10, 10^4\}$ independent of κ. A numerical integration shows that the integral in (5) indeed is

[1] This assumption is justified by the fact that electric and magnetic effects decouple in the low-frequency limit, *cf.*, the right hand side of (1).

[2] The corresponding magnetic elements elements are [7]

$$t_{1\ell}(\kappa) = -\frac{\mathrm{j}_\ell(\kappa)(\kappa\sqrt{\epsilon(\kappa)}\mathrm{j}_\ell(\kappa\sqrt{\epsilon(\kappa)}))' - (\kappa \mathrm{j}_\ell(\kappa))'\mathrm{j}_\ell(\kappa\sqrt{\epsilon(\kappa)})}{\mathrm{h}_\ell^{(1)}(\kappa)(\kappa\sqrt{\epsilon(\kappa)}\mathrm{j}_\ell(\kappa\sqrt{\epsilon(\kappa)}))' - (\kappa \mathrm{h}_\ell^{(1)}(\kappa))'\mathrm{j}_\ell(\kappa\sqrt{\epsilon(\kappa)})}.$$

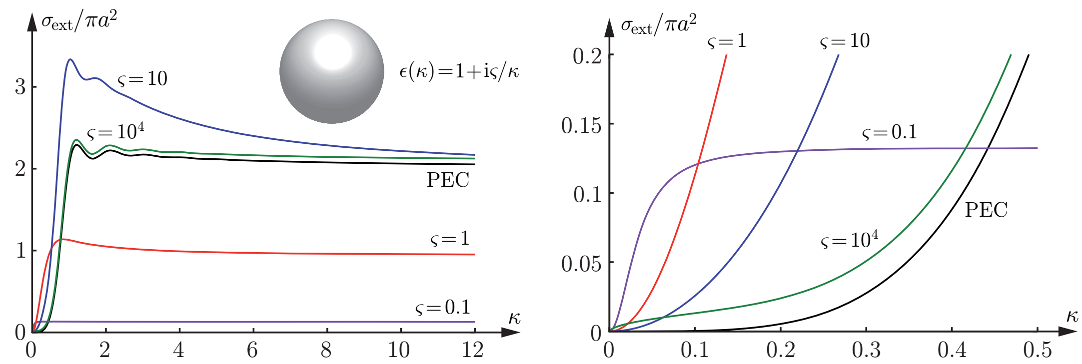

Figure 1: The extinction cross section in units of πa^2 for a homogeneous and isotropic sphere with $\epsilon' = 1$. The right figure is a close-up of the left figure for low frequencies, and as a consequence of (5), all curves (except for the one labeled PEC) have the same value of the integral when weighted with $1/\kappa^2$.

equal to 2π (within relative errors less than 1% by integrating up to $\kappa = 34$) for the four curves in Fig. 1. The fifth curve marked by PEC is discussed in Sec. 3, and it can be shown to this curve has an integral which is π rather than 2π when weighted with $1/\kappa^2$. At a first glance, one might not expect that the four curves for the static conductivity have the same integral of $\nu(\kappa)/\kappa^2$. However, the left figure in Fig. 1 makes it plausible that the curves with a low value of ς are shifted toward lower frequencies in such a manner that the integral is preserved. This behavior can further be understood by examining the zeros of the nominator in (3) in the vicinity of the negative imaginary axis in the complex κ-plane.

The puzzling identity (5) is partially explained by rejecting the conductivity model in the low-frequency limit. For example, this is done by introducing a frequency dependent static conductivity $\varsigma = \varsigma(\kappa)$ for sufficiently low frequencies, or by a regularization (*i.e.*, replacing $\mathrm{i}\varsigma/\kappa$ by $\mathrm{i}\varsigma/(\kappa + \mathrm{i}\varepsilon)$ for $\varepsilon > 0$) of the static conductivity model in the low-frequency limit. This is analogous to the procudeure of letting the relaxation time τ in the Debey model approach infinity. Alternatively, one may transform (5) into an inequality by removing a portion of the integral in the neighborhood of $\kappa = 0$, *viz.*, for any $\varepsilon > 0$,

$$\int_\varepsilon^\infty \frac{\nu(\kappa)}{\kappa^2}\,\mathrm{d}\kappa \leq 2\pi. \tag{6}$$

The left hand side of (6) now depends on both ε and ς, and generally it is no longer true that the integral is equal to 2π.

3 A comparison with the PEC boundary condition

The transmission boundary conditions associated with (3) are formulated such that the tangential components of the electric and magnetic fields are continuous over the boundary surface. The corresponding PEC boundary condition states that the tangential component of the electric field vanishes on the target. For a fixed $\kappa > 0$, (3) approaches the transition matrix element for the PEC boundary condition as the magnitude of ϵ (for example, by letting $\varsigma \to \infty$ in the presence of a static conductivity term) tends to infinity independent of μ. This is generally not true in the low-frequency limit as it is required that the index of refraction $\sqrt{\epsilon(\kappa)\mu(\kappa)}$ should be bounded as $\kappa \to 0$. Instead, the PEC boundary condition is obtained in the low-frequency limit by simultaneously letting $\epsilon(0) \to \infty$ and $\mu(0) \to 0$ as suggested by the discussion in Ref. 1.

4 The effects of diamagnetic material parameters in the low-frequency limit

Under the assumption that diamagnetic materials exist with $\mu(0)$ less than unity, at a first glance it seems to contradict (1) since the second term on the right hand side then becomes negative [9, p. 283]. However, the negative value of $\mu(0)$ is compensated by a positive value of $\epsilon(0)$ as seen below. Use the fact that $\kappa\epsilon(\kappa)$ and $\kappa\mu(\kappa)$ are Herglotz functions, *i.e.*, holomorphic functions in the upper half of the complex κ-plane, there satisfying $\mathrm{Im}\,\kappa\epsilon(\kappa) \geq 0$

and $\operatorname{Im} \kappa\mu(\kappa) \geq 0$ [4]. Now, $\kappa(n(\kappa) - n_\infty)$ defines a new Herglotz function, where $n_\infty = \lim_{\kappa\to\infty} n(\kappa)$ and $n(\kappa) = \sqrt{\epsilon(\kappa)\mu(\kappa)}$ denotes the index of refraction. Hence, $n(0)$ is bounded from below by n_∞, and from the inequality between the geometric and arithmetic means, one have (equality on the left hand side of (7) if and only if $\epsilon(0) = \mu(0)$) [10, pp. 16–18]

$$\frac{\epsilon(0) + \mu(0)}{2} \geq \sqrt{\epsilon(0)\mu(0)} \geq n_\infty. \tag{7}$$

Now, since the relativistic causality condition postulates that no signal can propagate with a phase velocity greater than the phase velocity in free space, it is concluded that n_∞ is bounded from below by unity (alternatively, one may use the Kramers-Kroing relations; like ϵ_∞ and μ_∞, also n_∞ is non-unique from a modeling point of view), and (7) yields that $\epsilon(0) + \mu(0) \geq 2$. Under the assumption that $\epsilon(0)$ is positive (or more generally, not less than $\mu(0)$), the parenthesis on the right hand side of (1) can be estimated from below by $(\epsilon(0) + \mu(0) - 2)/(\epsilon(0) + 2)$ which indeed is non-negative. It is thus concluded that the right hand side of (1) is non-negative in the presence of diamagnetic effects in the low-frequency limit, and there is no contradiction with the non-negative nature of the extinction cross section.

5 Conclusions

It is concluded that the extinction cross section integrated over all frequencies is independent of the choice of static conductivity ς as long as it is non-zero. The PEC boundary condition is obtained in the low-frequency limit by letting not only the magnitude of $\epsilon(0)$ approach infinity, but simultaneously sending $\mu(0)$ to zero such that $n(0) = \sqrt{\epsilon(0)\mu(0)}$ is well-defined. Furthermore, diamagnetic material parameters in the low-frequency limit cause no problem in (1) since any $\mu(0)$ less than unity is compensated by a positive $\epsilon(0)$ such that the right hand side of (1) remains non-negative.

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