

Stability and causality of effective material parameters for biased ferromagnetic materials

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Abstract

We show that the small signal permeability derived from a linearization of the Landau-Lifshitz-Gilbert equation describing a ferromagnetic particle is unstable. Stability is recovered when the field external to the particle is considered as input signal. When calculating the effective permeability of a composite material, consisting of aligned, biased single domain particles, the result is a stable, and hence causal, material.

1 Introduction

Magnetism is one of the earliest documented electromagnetic phenomena, one of the first references being Plato [1]. In ferromagnetic media, there is a strong coupling between the magnetic moments in neighboring atoms. The precise mechanism of this coupling still remains obscure, but a phenomenological model based on the time evolution of a magnetic moment in a magnetic field was presented in 1935 by Landau and Lifshitz [2]. Their model has subsequently been augmented, and in particular Gilbert has refined the phenomenological model of losses [3]. In this paper, we discuss some consequences of linearizing this equation to obtain a small signal permeability, which is often done in the modelling of ferrites.

2 Physical description

In a ferromagnetic material, each atom has a magnetic moment which tend to be aligned with each other. In a mesoscale model, we consider the magnetic moment per unit volume, *i.e.*, the magnetization \mathbf{M} . This is described by the Landau-Lifshitz-Gilbert model [2, 3]

$$\frac{\partial \mathbf{M}}{\partial t} = -\gamma \mu_0 \mathbf{M} \times \mathbf{H}_{\text{eff}} + \alpha \frac{\mathbf{M}}{M_s} \times \frac{\partial \mathbf{M}}{\partial t} \quad (1)$$

Since the right hand side is orthogonal to \mathbf{M} , it is immediately seen that the amplitude of the magnetization is preserved, $|\mathbf{M}| = M_s$, where M_s is the saturation magnetization. The first term, where $\gamma = 1.76 \cdot 10^{11}$ C/kg is the gyromagnetic ratio and μ_0 is the permeability of vacuum, corresponds to the torque on a magnetic moment in a magnetic field, which typically produces a precessional motion of the magnetization around an axis parallel to the effective magnetic field \mathbf{H}_{eff} . The second term is the phenomenological damping term suggested by Gilbert, which describes the tendency of the magnetization to eventually align with the field \mathbf{H}_{eff} . The loss parameter is typically $\alpha \approx 0.1$.

The effective field \mathbf{H}_{eff} has many contributions, some of which do not belong to classical electromagnetism but rather to quantum mechanics. The most common and strong effects are

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + \mathbf{H}_{\text{an}} + \mathbf{H}_{\text{ex}} + \mathbf{H}_{\text{me}} \quad (2)$$

where \mathbf{H} is the classical magnetic field, which appears in Maxwell's equations. By including the possibility of a demagnetization contribution in this field (which is sometimes called shape anisotropy), the anisotropy field \mathbf{H}_{an} is

only due to the atomic lattice of the material. For some materials, a linear model is sufficient for this field, but it is in general a nonlinear function of the direction of the magnetization. The exchange field \mathbf{H}_{ex} in its turn, models the strong alignment and formation of magnetic domains in the material. It is often written $\mathbf{H}_{\text{ex}} = l_{\text{ex}}^2 \nabla^2 \mathbf{M}$, where the exchange length l_{ex} is of the order of a few nanometers. Finally, the magnetoelastic field \mathbf{H}_{me} models the possibility that mechanical strain may affect the magnetization of the material.

3 Single domain particles

The different terms in the effective field \mathbf{H}_{eff} contribute with different strengths depending on the situation. When the magnetization varies on a nanometer scale, the exchange field becomes very strong and therefore the material tends to align its magnetic moment locally in order to reduce the influence of $\nabla^2 \mathbf{M}$. If a magnetic specimen is small enough, say 10–100 nm, it is energetically favorable to align *all* of the atoms in the particle, except for a thin boundary layer. This is the spontaneous formation of a single magnetic domain in nanosized particles [4].

For describing single domain particles, the Landau-Lifshitz-Gilbert equation (1) is very suitable, since the situation is strongly idealized. For spherical (or more generally, spheroidal) particles, the classical magnetic field can be written (due to the classical solution of a spherical particle in a homogeneous external field being a homogeneous internal field and an external dipole field) $\mathbf{H} = \mathbf{H}^e - \mathbf{N}_d \mathbf{M}$, where \mathbf{H}^e is the magnetic field external to the particle, and \mathbf{N}_d is the demagnetization tensor of the particle. For spherical particles, we have $\mathbf{N}_d = \mathbf{I}/3$. Since the saturation magnetization is very strong, this shows that the internal field in a single domain particle is usually very strong, even in the absence of an external field. In the following, we describe the delicate interplay between the external bias field and the demagnetization field.

4 Small signal model

When the magnetic specimen is subjected to a (relatively weak) time-varying magnetic field, we assume the fields can be written

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1 e^{-i\omega t}, \quad \mathbf{M} = \mathbf{M}_0 + \mathbf{M}_1 e^{-i\omega t}, \quad \mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{eff}0} + \mathbf{H}_{\text{eff}1} e^{-i\omega t} \quad (3)$$

where the fields with index 0 indicates static quantities. For simplicity, we ignore all effects except demagnetization and linearize the Landau-Lifshitz-Gilbert equation (1) around the static fields. This leads to the gyrotropic relation [5]

$$\mathbf{M}_1 = \frac{1}{(\beta - i\alpha\omega/\omega_s)^2 - (\omega/\omega_s)^2} \left((\beta - i\alpha\omega/\omega_s) \mathbf{I} + i(\omega/\omega_s) \mathbf{m}_0 \times \right) \mathbf{H}_1 = \chi(\omega) \mathbf{H}_1 \quad (4)$$

where $\beta = |\mathbf{H}_0^e|/M_s - 1/3$ is the bias parameter, $\omega_s = \gamma\mu_0 M_s$ is the intrinsic precession angular frequency, and $\mathbf{m}_0 = \mathbf{M}_0/M_s$ is the direction of the zeroth order magnetization. The term $-1/3$ in the expression for β corresponds to the demagnetization tensor \mathbf{N}_d . Since the relation between the magnetization and the magnetic field is gyrotropic with the axis of gyrotropy being equal to the static magnetization, the natural waves propagating along the direction of magnetization are left and right hand circularly polarized, with effective permeabilities

$$\mu_{\pm} = 1 + \frac{1}{\beta + (\pm 1 - i\alpha)\omega/\omega_s} \quad (5)$$

Magnetic losses are proportional to $\omega \text{Im}(\mu_{\pm})$, and it is readily verified that this number is positive for all real ω . But this does not mean the material is neither passive, causal, or stable, since the union of these properties require that all poles of $\boldsymbol{\mu} = \mathbf{I} + \boldsymbol{\chi}$, and hence all poles of its eigenvalues μ_{\pm} , are in the lower half complex ω plane. As we shall see, this is no obvious thing.

5 Causality and stability

Causality and stability are two important system aspects, and are usually studied in terms of the location of the poles of the system transfer function. In our case, the transfer function is the susceptibility $\chi(\omega)$. The criterion for this representing a causal system is that all poles are in a lower half plane, *i.e.*, there is an upper bound on the imaginary

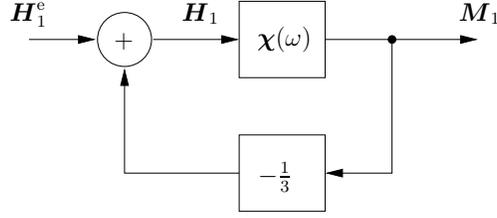


Figure 1: A system with negative feedback, corresponding to using a model where the external magnetic field is used as input. Even though the susceptibility $\chi(\omega)$ is unstable with respect to its input \mathbf{H}_1 , the negative feedback makes the relation between the external field \mathbf{H}_1^e and the magnetization \mathbf{M}_1 stable.

part. The system is stable if this upper bound is negative, *i.e.*, stability implies causality. The poles are given by setting the denominator in (5) to zero, implying

$$\omega_{\pm} = \frac{-\beta(\pm 1 + i\alpha)}{1 + \alpha^2} \quad (6)$$

i.e., the imaginary part of the pole is $-\beta\alpha/(1 + \alpha^2)$. Since $\beta = |\mathbf{H}_0^e|/M_s - 1/3$ is negative for a small external field, the pole can very well be in the upper half plane, which means the impulse response can be exponentially increasing in time. This violates stability but not causality. When considering causality and stability, one must be careful to choose an *output* which should be related to a suitable *input*. The output should obviously be the magnetization \mathbf{M}_1 , but the proper input could be both the internal field \mathbf{H}_1 and the external field \mathbf{H}_1^e , and these are related via the demagnetization factor $1/3$ according to $\mathbf{H}_1 = \mathbf{H}_1^e - \mathbf{M}_1/3$. Inserting this expression in (4), implies

$$\mathbf{M}_1 = \chi(\omega) \left(\mathbf{H}_1^e - \frac{\mathbf{M}_1}{3} \right) \implies \mathbf{M}_1 = \frac{\left((\beta + 1/3 - i\alpha\omega/\omega_s)\mathbf{I} + i(\omega/\omega_s)\mathbf{m}_0 \times \right) \mathbf{H}_1^e}{(\beta + 1/3 - i\alpha\omega/\omega_s)^2 - (\omega/\omega_s)^2} \quad (7)$$

The first part of this equation can be interpreted as a system with negative feedback as depicted in Figure 1. It is seen that the poles of the transfer function between \mathbf{H}_1^e and \mathbf{M}_1 are

$$\omega_{\pm} = \frac{-(\beta + 1/3)(\pm 1 + i\alpha)}{1 + \alpha^2} \quad (8)$$

and since $\beta + 1/3 = |\mathbf{H}_0^e|/M_s - 1/3 + 1/3 = |\mathbf{H}_0^e|/M_s > 0$, we see that this transfer function is stable, and therefore causal. Thus, the small signal susceptibility $\chi(\omega)$ can not be deemed unphysical just on account of its poles, which simply reflects the choice of input signal.

6 Homogenization

We now turn to the question of what happens when the small signal susceptibility $\chi(\omega)$ is used in the calculation of effective material parameters for a composite material. We assume the composite material is of infinite extension, where an idealized microgeometry consisting of aligned single domain magnetic particles embedded in a nonmagnetic background material is shown in Figure 2.

The first effect of this configuration is to change the static magnetic field acting on each particle. If the volume fraction of particles is f_1 and each particle has magnetization $\mathbf{M}_0 = M_s \mathbf{m}_0$, then the effective static magnetization is $f_1 \mathbf{M}_0$. If the composite material is subjected to an externally controlled bias magnetic field \mathbf{H}_0^e , then the field at one particle is $\mathbf{H}_0^{e'} = \mathbf{H}_0^e + f_1 \mathbf{M}_0/3$, see for instance [6, p. 162]. This implies the bias parameter β in this case is

$$\beta = \frac{|\mathbf{H}_0^{e'}|}{M_s} - \frac{1}{3} = \frac{|\mathbf{H}_0^e|}{M_s} + \frac{f_1}{3} - \frac{1}{3} = \frac{|\mathbf{H}_0^e|}{M_s} - \frac{f_2}{3} \quad (9)$$

where $f_2 = 1 - f_1$ is the volume fraction of the surrounding material. The effective relative permeability can be calculated using the following generalized Maxwell-Garnett (or Hashin-Shtrikman) formula [7, p. 145]

$$\boldsymbol{\mu}_{\text{eff}} = \mathbf{I} + f_1(\boldsymbol{\mu}_1 - \mathbf{I})[\mathbf{I} + (f_2/3)(\boldsymbol{\mu}_1 - \mathbf{I})]^{-1} \quad (10)$$

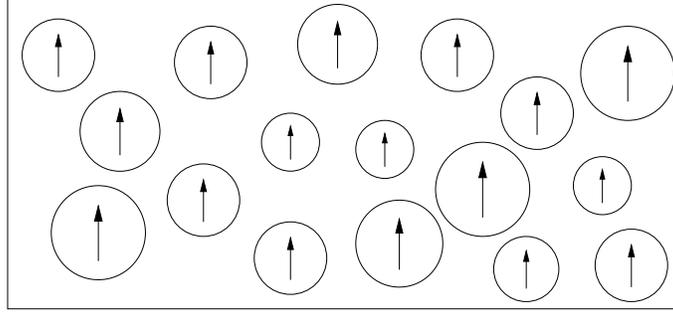


Figure 2: Idealized microgeometry of the composite material. Spherical particles, possibly with varying size but having the same magnetization, are dispersed in a nonmagnetic background material.

It can be shown that the poles of the effective permeability calculated according to this formula using $\mu_1(\omega) = \mathbf{I} + \chi(\omega)$ are [8]

$$\omega_{\pm} = \frac{-(\beta + f_2/3)(\pm 1 + i\alpha)}{1 + \alpha^2} \quad (11)$$

Since β in the composite material is given by (9), we see that these poles lie in the lower half plane, and thus the effective material properties of the composite material correspond to a stable, and thus causal, material. It is also seen that the poles do not depend on the volume fraction, only on the material parameters of the ferromagnetic particles and the external bias field \mathbf{H}_0^e .

7 Conclusions

We have demonstrated that the stability and causality of the small signal material parameters describing single domain ferromagnetic particles depend on which field is taken as the input. When the Landau-Lifshitz-Gilbert equation, which models the full dynamics of the magnetization, is linearized, the relation between the internal magnetic field and the magnetization is unstable. Though surprising at first sight, the explanation is due to the fact that the internal field is not independent of the magnetization, and stability and causality is recovered when taking the external field as input. When the effective material permeability of a composite material is calculated with a standard mixing formula, the result is shown to be stable and causal, in spite of the fact that the calculation was formally made using unstable susceptibilities describing the particles.

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