The Electrodynamics of Rotating Slow-Light Waveguides with Structural Disorder – Optical Gyroscopes, Degeneracy Splitting, and the Creation of a Dead-Zone

***Ben Z. Steinberg* and Jacob Scheuer**

School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978 Israel
steinber@eng.tau.ac.il, kobys@eng.tau.ac.il

Abstract

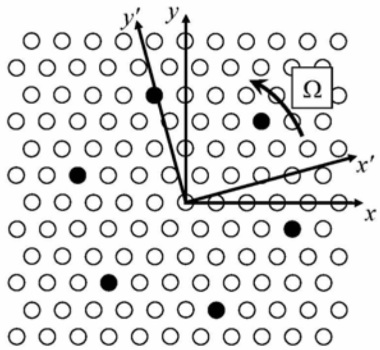
The effect of structural disorder on the electrodynamics of slowly and rigidly rotating coupled resonators circular waveguide is studied, in the rotating system rest-frame. When stationary and ideal, this waveguide supports degenerate modes. It is shown that slow rigid rotation as well as structural disorder can be viewed as two competing symmetry-breaking events. Hence both cause splitting of mode degeneracy. The system can be used as a gyroscope as long as the symmetry breaking due to rotation is stronger than that due to disorder. This causes a creation of a dead-zone in the gyroscope operation curve.

1. Introduction

An electromagnetic wave that propagates along a slowly rotating circular path, and observed in the rotating path rest-frame, accumulates an additional phase that depends linearly on the path angular velocity Ω. This phase shift, known as the Sagnac effect [1], has been studied quite extensively in the literature with interest that stems from theoretical as well as practical points of view; highly accurate gyroscopes were built using this effect [2]. The classical early studies, however, were restricted to optical signals that are described as local plane waves (or fiber-modes), their propagation path is homogeneous and can be defined geometrically. Then the Sagnac effect is expressed as $\Delta\phi = 4N\omega\Omega a / c^2$ where ω is the frequency, a is the area enclosed by the path consisting of N turns, and c is the vacuum speed of light [2]. This ***century-old result*** establishes the area a as the ***fundamental factor limiting the responsivity of Sagnac effect***. Also, due to this result it has been believed that the medium properties do not influence the Sagnac effect [2]. These observations put ***harsh technological bottlenecks*** in the endeavor to miniaturize optical gyroscopes: chip-scale devices have limited area, and are restricted to a single turn.

Recently, the Sagnac effect was studied in slow-wave structures consisting of single and multi mode coupled micro-cavities [3-4]. In [3] we investigated Electrodynamics (ED) of rotating photonic-crystal based gyroscope (Fig. 1). The light path is defined by an array of weakly coupled resonators, and constitutes a slow-light waveguide (known also as CROWs: Coupled Resonator Optical Waveguides). It has been shown that, unlike traditional perceptions, the dielectric structure can significantly affect the Sagnac phase shift. In [4] the rotation of CROWs composed of ring-resonators was studied, showing a potential for significant enhancement of the Sagnac effect. Subsequently, we studied the ED of a general rotating micro-cavity with mode degeneracy [5], and the ED of a CROW consisting of such cavities [6]. It has been shown that under rotation, CROWs with mode degeneracy supports the *emergence of a periodic super-structure* that modifies the array properties, and opens a forbidden frequency gap in the center of the CROW transmission curve, lead to exponential sensitivity to rotation [6]. These studies expose new physical effects associated with the ED of rotating micro-structures and may offer a way to break the century-old limitation of sensitivity that depends only on the loop area.

In any realization of these new effects, structural disorder (e.g. due to limited fabrication accuracy) may introduce new factors that are not present in the idealized geometries. Hence, the purpose of the present work is to investigate the effect of structural disorder on the ED of rotating CROWs. We concentrate first on the CROWs consisting of micro-cavities that do not support mode degeneracy. Hence, our study applies to the basic schemes shown in Fig. 1. Note, however, that although each individual micro-cavity does not support mode-degeneracy, the entire ideal structure does; at any given frequency within its transmission band and in the absence of structural disorder, it supports at least two optical signals: a clock-wise (CW) and a counter-clock-wise (CCW) propagating mode. This fact will be exploited in our analysis.



(a)

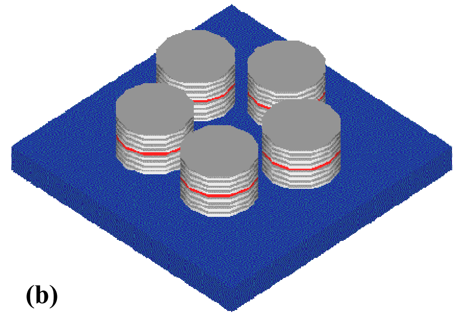

(b)

Figure 1. (a) The Photonic-Crystal based slow-light optical gyroscope. The crystal is made of dielectric cylinders. Filled circles represent weakly-coupled micro-resonators created by local defects. (b) A realization of a CROW based optical Gyro using vertical cavity surface emitting lasers. This scheme is more appropriate for gain inclusion.

2. Formulation

Under stationary conditions (Ω=0), the *entire ideal structure* possesses second order mode degeneracy at each of its optical resonances; it supports two modes propagating in CW and CCW directions. When the structure rotates, the degenerate resonance splits into two distinct resonances [5]. Similarly, structural disorder splits mode degeneracy at stationary conditions [7]. Thus, we invoke the theory of mode-degeneracy under rotation [5], and extend it to hold also under structure inaccuracy using the tools of [7]. This combined approach yields the effect of rotation *and* disorder on the mode splitting. In addition, it reveals an interesting picture of two competing effects: both disorder and rotation break mode degeneracy and cause splitting. Splitting due to the former is Ω-independent, while splitting due to the latter is linear with Ω. The device can be used as a gyro only when the latter exceeds the former. Hence, a dead-zone in Ω is formed.

Our starting point is the set of Maxwell's equations governing the ED of slowly rotating systems in their rest frame [8-9]. Their form is identical to that of the conventional set of Maxwell's equations,

$$\nabla\times\vec{E}=i\omega\vec{B},\quad \nabla\times\vec{H}=-i\omega\vec{D},\quad \nabla\cdot\vec{B}=0,\quad \nabla\cdot\vec{D}=0, \tag{1.1}$$

while rotation is manifested only via the more elaborated constitutive relations

$$\vec{D}=\varepsilon\vec{E}-c^{-2}(\vec{\Omega}\times\vec{r})\times\vec{H},\quad \vec{B}=\mu\vec{H}+c^{-2}(\vec{\Omega}\times\vec{r})\times\vec{E}. \tag{1.2}$$

Here ε,μ are the stationary material properties, c is the vacuum speed of light, and $\vec{\Omega}=\hat{z}\Omega$ is the medium angular velocity. A wave equation is obtained directly from Eqs. (1.1)-(1.2), by applying the standard procedure in conjunction with a slow rotation assumption (keeping terms up to first order in Ω),

$$\Theta\vec{H}=k^2\vec{H}+ik\mathbf{L}_\Omega\vec{H},\quad k=\omega/c,\quad \Theta\triangleq\nabla\times\varepsilon_r^{-1}(\vec{r})\nabla\times \tag{1.3}$$

where ε_r is the relative dielectric property, and $\mathbf{L}_\Omega$ is the rotation operator defined by

$$\mathbf{L}_\Omega\vec{H}=\nabla\times\varepsilon_r^{-1}(\vec{r})\vec{\beta}(\vec{r})\times\vec{H}+\varepsilon_r^{-1}(\vec{r})\vec{\beta}(\vec{r})\times\nabla\times\vec{H},\quad \vec{\beta}=\vec{\Omega}\times\vec{r}/c. \tag{1.4}$$

Let $\vec{H}_\Omega$ be the magnetic field of the entire structure, under rotation and structure variation. It satisfies the wave equation (1.3)-(1.4), with $\varepsilon_r(\vec{r})$ describing now the entire dielectric structure, including structural disorder. At rest, and in the absence of structural disorder, the two degenerate modes $\vec{H}_0^{(m)}(\vec{r})$ of the *entire* structure satisfy:

$$\Theta_0\vec{H}_0^{(m)}=k_U^2\vec{H}_0^{(m)},\quad \Theta_0\equiv\nabla\times\left[1/\varepsilon_U(\vec{r})\right]\nabla\times,\quad k_U=\omega_U/c,\quad m=1,2 \tag{1.5}$$

where $\varepsilon_U(\vec{r})$ describes the entire *unperturbed* structure and ω_0 is the corresponding resonance frequency. As will be shown, rotation and structural disorder cause it to split into two different resonances. Since Θ_0 is self-adjoint $\vec{H}_0^{(m)}(\vec{r})$ can always be presented as real and orthogonal set. Also, we write:

$$\Theta=\Theta_0+\delta\Theta,\quad \delta\Theta=\nabla\times\left(\delta\left[1/\varepsilon(\vec{r})\right]\right)\nabla\times,\quad \delta\left[1/\varepsilon(\vec{r})\right]=\varepsilon^{-1}(\vec{r})-\varepsilon_U^{-1}(\vec{r}) \tag{1.6}$$

so the wave equation (1.3)-(1.4) can be written as [neglecting higher order terms proportional to $\Omega\delta(1/\varepsilon)$]:

$$\Theta_0\vec{H}_\Omega - k^2\vec{H}_\Omega = (ik\mathbf{L}_\Omega - \delta\Theta)\vec{H}_\Omega. \tag{1.7}$$

Clearly, the LHS of (1.7) is nothing but the conventional wave equation governing the field in the stationary and unperturbed structure. The RHS describes two effects; that of rotation and that of structural disorder. We follow now the steps outlined in [5] with the only exception that an additional term (structural disorder) appears in the RHS. The total field is a summation over $\vec{H}_0^{(n)}(\vec{r})$ serving merely as building blocks:

$$\vec{H}_\Omega = \sum\nolimits_{n=1}^{2} a_n \vec{H}_0^{(n)} \tag{1.8}$$

Now we substitute into (1.7), use (1.5), perform inner product with $\vec{H}_0^{(m)}$ and use their mutual orthogonality. The result is the 2X2 matrix-eigenvalue equation for the expansion coefficients a_n and frequency splitting $k-k_U$,

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(k-k_U\right)2k_U\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{1.9}$$

where

$$D_{mn} = -ik_U(\mathbf{L}_\Omega\vec{H}_0^{(n)},\vec{H}_0^{(m)}) + (\delta\Theta\vec{H}_0^{(n)},\vec{H}_0^{(m)}) = k_U^2\Omega B_{mn} + d_{mn} \tag{1.10}$$

and where *(f, g)* is the L_2 inner-product for vector fields [5]. The two different eigenvalues of the equation above represent the frequency splitting. B_{nm} here were studied in detail in [5]. They depend only on the perfect (unperturbed) structure and form a skew-symmetric imaginary matrix, with $B_{11}=B_{22}=0$. d_{mn} depend only on $\delta\Theta$ and represent only structure disorder effects. One can show that $d_{12}=d_{21}$ and $d_{11}+d_{12}\rightarrow 0$ as the number of (statistically independent) cavities comprising the structure increases. Solving algebraically for the eigenvalues of (1.9) we get

$$k(\Omega) - k_U = \pm(k_U/2)\left|B_{12}\right|\sqrt{\Omega_d^2+\Omega^2}, \quad \Omega_d = \left(d_{12}^2 - d_{11}d_{22}\right)^{1/2} \Big/ k_U^2\left|B_{12}\right|. \tag{1.11}$$

Thus, the splitting *vs.* Ω possesses the form of Hyperbola, with a “dead-zone” Ω_d originating from the “competition” between disorder-induced splitting and rotation-induced splitting, as described above. The d_{mn} coefficients, responsible for the dead-zone extent, are related to the variance of the individual cavity resonance frequency. This is shown by expanding each of the (doubly-degenerate) modes of the entire unperturbed structure $\vec{H}_0^{(m)}(\vec{r})$, with the modes $\vec{H}_j(\vec{r}) = \vec{H}_0(\vec{r}-\vec{r}_j)$ of the M isolated micro-cavities in the structure (consistent with tight-binding theory), and with expansion coefficients c_j that should satisfy the CROW dispersion relation [10]

$$\vec{H}_0^{(m)} = \sum\nolimits_{j=1}^{M} c_j^m(\omega)\vec{H}_j(\vec{r}), \quad m=1,2. \tag{1.12}$$

By definition, the modal field of the *j*-th cavity, $\vec{H}_j(\vec{r})$, satisfies

$$\Theta^j H_j = k_{00}^2 H_j, \quad k_{00} = \omega_{00}/c, \quad \Theta^j = \nabla\times(1/\varepsilon_r^j)\nabla\times, \quad j=1,\ldots,M \tag{1.13}$$

and where ω_{00} is the resonance frequency of the (identical, unperturbed) micro-cavities, and ε_r^j is the relative dielectric property of the *j*-th cavity ($\varepsilon_r^j,\varepsilon_r^k$ differ only by a shift of coordinates). Substituting (1.12) into the d_{mn} definition (1.10), using (1.13), the fact that $\nabla\times H_j = -i\omega_{00}\varepsilon_0\varepsilon_r^j E_j$ and that E_j is highly localized within the *j*-th micro-cavity, we end up with

$$d_{mn} = \omega_{00}^2\varepsilon_0^2\sum\nolimits_{j=1}^{M} c_j^n c_j^m\left(\delta\varepsilon_r^j\vec{E}_j,\vec{E}_j\right) = 2k_{00}c^{-1}\left\|H_0\right\|^2\sum\nolimits_{j=1}^{M} c_j^n \overline{c}_j^m\delta\omega_j \tag{1.14}$$

where the last equality above is due to cavity perturbation theory[11], and $\delta\omega_j$ is the deviation of the *j*-th micro-cavity resonant frequency due to structural disorder. Hence one can relate the statistics of the individual resonances variations to the dead-zone extent.

3. Example and Conclusions

We simulated a 7- active cavity (micro-lasers) CROW gyro. When stationary, the ideal structure has 4 resonances, 3 of which are doubly degenerate. Their splitting under rotation is shown in Fig. 2a. Fabrication errors are modeled by introducing variations in the resonance frequencies of the individual cavities, normally distributed with variance σ. Fig. 2b shows the splitting for $\sigma = 10^{-3}$. The disorder splits mode degeneracy, even without rotation, hence reducing the responsivity at low rotation rates. The excellent agreement with the hyperbolic fit predicted by our preliminary analysis is evident. Our theory enables to estimate the responsivity and the "dead-zone" – the smallest detectable rotation rate.

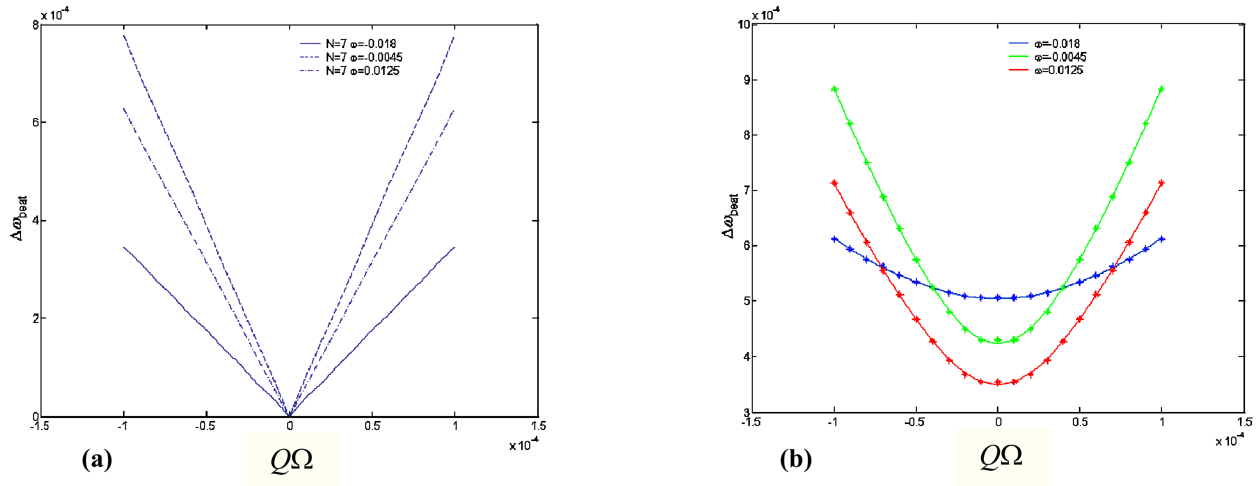


Fig. 2 (a) Splitting of the ideal CROW with 7 micro-cavities, vs. $Q\Omega$ (Q is the ideal CROW rotation-coupling coefficient [3]). (b) Splitting vs. rotation under structural disorder. Numerical results are shown by starts, and solid lines show hyperbolic fit.

References

[1] G. Sagnac, *C.R. Acad. Sci.,* **95,** 708 (1913).

[2] H. J. Arditty and H. C. Lefevre, "Sagnac Effect in Fiber Gyroscopes," *Opt. Lett.* **6(8),** 401 (1981).

[3] Ben Z. Steinberg, "Rotating Photonic Crystals: A Medium for Compact Optical Gyroscopes," *Phys. Rev. E*, **71**, 056621 (2005).

[4] J. Scheuer and A. Yariv, "Sagnac effect in coupled resonator slow light waveguide structures," *Phys. Rev. Lett.* **96**, 053901 (2006).

[5] B. Z. Steinberg and A. Boag, "Splitting of Micro-Cavity Degenerate Modes in Rotating Photonic Crystals - The Miniature Optical Gyroscopes," *J. Opt. Soc. Am. B*, **24(1)**, 142 (2007).

[6] B. Z. Steinberg, J. Scheuer, and A. Boag, "Rotation Induced Super Structure in Slow-Light Waveguides with Mode Degeneracy: Optical Gyroscopes with Exponential Sensitivity," *J. Opt. Soc. Am. B*, **25(5)**, 1216 (2007).

[7] B. Z. Steinberg, A.Boag, and R. Listitsin, "Sensitivity analysis of narrowband photonic crystal filters and waveguides to structure variationas and inaccuracy," *J. Opt. Soc. Am. A.* **20(1)**, 138 (2003).

[8] J. L. Anderson and J. W. Ryon, "Electromagnetic Radiation in Accelerated Systems," *Physical Review* **181(5)**, 1765-1775 (1969).

[9] T. Shiozawa, "Phenomenological and Electron-Theoretical Study of the Electrodynamics of Rotating Systems," *Proc. IEEE*, **61(12)**, 1694-1702 (1973).

[10] A. Boag, and B. Z. Steinberg, "Narrow Band Micro-Cavity Waveguides in Photonic Crystals," *J. Opt. Soc. Am. A*, **18(11)**, 2799-2805,(2001).

[11] R. F. Harrington, *Time Harmonic Electromagnetic Fields,* McGraw Hill, New York (1961).