A general approach for deriving bounds in electromagnetic theory

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Abstract

This paper reports on a systematic procedure for deriving physical bounds in electromagnetic theory. The approach is based on the holomorphic properties of certain Herglotz functions and their asymptotic expansions in the low- and high-frequency regimes. A family of integral identities or summation rules is obtained with values governed by the coefficients in the low- and high-frequency expansions. In particular, summation rules for plane-wave scattering by a homogenous and isotropic sphere is derived and verified numerically by computing the extinction cross section and the bistatic radar cross section in the forward direction. It is concluded that summation rules based on Herglotz functions show great potential for deriving new physical bounds in, *e.g.*, scattering and antenna problems.

1 Introduction

Summation rules, or more generally dispersion relations, is a promising technique in theoretical physics for analyzing particle collisions and scattering problems of acoustic, electromagnetic and elastic waves [1]. There are at least three main advantages of summation rules in the modeling of wave interaction with matter: i) they provide a consistency check of calculated quantities when the underlying mathematical model is known to be causal; ii) they may be used to verify whether a given mathematical model or an experimental outcome behaves causally or not; iii) they give rise to a priori bounds on quantities which are of experimental significance. In addition, summation rules are valuable as benchmark results from which to compare numerical solutions. Summation rule techniques have successfully been applied in Refs. 2–4 to a large class of scattering and antenna problems to establish physical bounds on the amount of interaction available in a given frequency interval. This paper, however, takes a more general approach and presents a systematic way of deriving new summation rules in electromagnetic theory.

2 Summation rules for Herglotz functions

Consider an arbitrary Herglotz function $h = h(\kappa)$ defined as a mapping from the upper half part of the complex κ-plane into itself, *i.e.*, let $h(\kappa)$ be a holomorphic function with the property that $\operatorname{Im} h(\kappa) > 0$ for $\operatorname{Im} \kappa > 0$ [5]. Assume that $h(\kappa)$ has the following low-frequency expansion in the vicinity of the origin:

$$h(\kappa) = a_{-1}\kappa^{-1} + a_1\kappa + \ldots + a_{2N-1}\kappa^{2N-1} + \mathcal{O}(\kappa^{2N}) \text{ as } \kappa \to 0, \tag{1}$$

where $a_{-n} = 0$ for $n = 2, 3, \ldots$. The corresponding asymptotic series at infinity with $b_m = 0$ for $m = 2, 3, \ldots$ reads

$$h(\kappa) = b_1\kappa + b_{-1}\kappa^{-1} + \ldots + b_{1-2M}\kappa^{1-2M} + \mathcal{O}(\kappa^{-2M}) \text{ as } \kappa \to \infty, \tag{2}$$

where we have used that $h(\kappa)/\kappa$ and $\kappa h(\kappa)$ are bounded as $\kappa \to \infty$ and $\kappa \to 0$, respectively [5]. The latter statement follows from the previous by observing that $-1/h(\kappa)$ defines a new Herglotz function whenever $h(\kappa)$ is a Herglotz function. The non-negative integers N and M are chosen such that they correspond to the first even monomials appearing in (1) and (2), respectively.

Assume $h(\kappa)$ satisfies the cross symmetry $h(\kappa) = -h^*(-\kappa^*)$ for complex-valued κ, implying that a_{2n-1} and b_{1-2m} are real-valued for $n, m = 0, 1, \ldots$ (in addition it is known that $-a_{-1}$ and b_1 are non-negative), and consider the integrals of $h(\kappa)/\kappa^\ell$ with respect to the contour in Fig. 1. Then, for any integer ℓ and $0 < \varepsilon < R$ we have

$$\int_{\varepsilon<|\kappa|<R} \frac{h(\kappa)}{\kappa^\ell}\, \mathrm{d}\kappa = \mathrm{i}\varepsilon^{1-\ell} \int_0^\pi h(\varepsilon \mathrm{e}^{\mathrm{i}\phi})\mathrm{e}^{\mathrm{i}(1-\ell)\phi}\, \mathrm{d}\phi - \mathrm{i}R^{1-\ell} \int_0^\pi h(R\mathrm{e}^{\mathrm{i}\phi})\mathrm{e}^{\mathrm{i}(1-\ell)\phi}\, \mathrm{d}\phi, \tag{3}$$

where κ is real-valued on the left-hand side of (3). Here, the small and large semicircles in Fig. 1 are parameterized by $\kappa(\phi) = \varepsilon \mathrm{e}^{\mathrm{i}(\pi-\phi)}$ and $\kappa(\phi) = R\mathrm{e}^{\mathrm{i}\phi}$, respectively, where $\phi \in [0, \pi]$. In order to evaluate the right-hand side of (3),

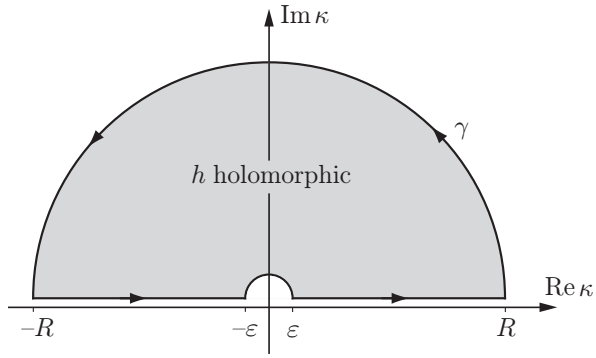

Figure 1: Trajectory γ in the complex κ-plane for the contour integral (3).

introduce the auxiliary integral

$$I_\ell = \int_0^\pi \mathrm{e}^{\mathrm{i}\ell\phi}\,\mathrm{d}\phi = \mathrm{i}\frac{1-\mathrm{e}^{\mathrm{i}\pi\ell}}{\ell} = \mathrm{i}\frac{2\alpha_\ell}{\ell}, \qquad \ell \neq 0,$$

with $I_0 = \pi$. Here, $\alpha_\ell = (1-(-1)^\ell)/2$, implying that $\alpha_\ell = 1$ for odd ℓ and $\alpha_\ell = 0$ for even ℓ. So, when (1) is inserted into (3) for a fixed $\varepsilon > 0$, the first integral on the right-hand side of (3) becomes

$$\begin{aligned}\mathrm{i}\varepsilon^{1-\ell}\int_0^\pi h(\varepsilon\mathrm{e}^{\mathrm{i}\phi})\mathrm{e}^{\mathrm{i}(1-\ell)\phi}\,\mathrm{d}\phi &= \mathrm{i}\left(a_{-1}\varepsilon^{-\ell}I_{-\ell} + \ldots + a_{2N-1}\varepsilon^{2N-\ell}I_{2N-\ell} + \mathcal{O}(\varepsilon^{2N+1-\ell})\right)\\ &= f(\varepsilon) + \mathcal{O}(\varepsilon^{2N+1-\ell}) + \mathrm{i}\pi\alpha_{\ell-1}a_{\ell-1}, \quad (4)\end{aligned}$$

where f is a real-valued function not necessarily continuous at $\varepsilon = 0$. Analogous to (4), for a fixed $R > 0$ the second integral on the right-hand side of (3) yields

$$\begin{aligned}-\mathrm{i}R^{1-\ell}\int_0^\pi h(R\mathrm{e}^{\mathrm{i}\phi})\mathrm{e}^{\mathrm{i}(1-\ell)\phi}\,\mathrm{d}\phi &= -\mathrm{i}\left(b_1R^{2-\ell}I_{2-\ell} + \ldots + b_{1-2M}R^{2-2M-\ell}I_{2-2M-\ell} + \mathcal{O}(R^{1-2M-\ell})\right)\\ &= f'(R) + \mathcal{O}(R^{1-2M-\ell}) - \mathrm{i}\pi\alpha_{\ell-1}b_{\ell-1}, \quad (5)\end{aligned}$$

where f' is a new real-valued function not necessarily continuous at infinity. Note that the rest terms $\mathcal{O}(\varepsilon^{2N+1-\ell})$ and $\mathcal{O}(R^{1-2M-\ell})$ in (4) and (5) are complex-valued in general.

From the cross symmetry $h(\kappa) = -h^*(-\kappa^*)$ it follows that $\mathrm{Re}\,h(\kappa) = -\mathrm{Re}\,h(-\kappa)$ and $\mathrm{Im}\,h(\kappa) = \mathrm{Im}\,h(-\kappa)$ for real-valued κ, and letting $\varepsilon \to 0$ and $R \to \infty$ in (3) therefore yields

$$\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{\varepsilon<|\kappa|<R}\frac{h(\kappa)}{\kappa^\ell}\,\mathrm{d}\kappa = 2\alpha_\ell\int_0^\infty\frac{\mathrm{Re}\,h(\kappa)}{\kappa^\ell}\,\mathrm{d}\kappa + 2\mathrm{i}\alpha_{\ell-1}\int_0^\infty\frac{\mathrm{Im}\,h(\kappa)}{\kappa^\ell}\,\mathrm{d}\kappa. \quad (6)$$

When $\varepsilon \to 0$ and $R \to \infty$, the rest terms $\mathcal{O}(\varepsilon^{2N+1-\ell})$ and $\mathcal{O}(R^{1-2M-\ell})$ in (4) and (5) vanish for $\ell = \ldots, 2N-1, 2N$ and $\ell = 2-2M, 3-2M, \ldots$, respectively. Thus, the left-hand side of (3) is well-defined if and only if $\ell = 2-2M, 3-2M, \ldots, 2N$. So, taking the imaginary part of (3) finally yields the following family of summation rules ($p = \ell/2$):

$$\int_0^\infty\frac{\mathrm{Im}\,h(\kappa)}{\kappa^{2p}}\,\mathrm{d}\kappa = \frac{\pi}{2}(a_{2p-1} - b_{2p-1}), \qquad p = 1-M, 2-M, \ldots, N, \quad (7)$$

where it has been used that odd ℓ vanish due to (6). Since $\mathrm{Im}\,h(\kappa)$ is non-negative, (7) can further be estimated from below by integrating over any finite region $K \subset [0,\infty)$ with center frequency $\kappa = \kappa_0$, *i.e.*,

$$B\kappa_0^{1-2p}\min_{\kappa\in K}\mathrm{Im}\,h(\kappa) + \mathcal{O}(B^3) \leq \int_K\frac{\mathrm{Im}\,h(\kappa)}{\kappa^{2p}}\,\mathrm{d}\kappa \leq \frac{\pi}{2}(a_{2p-1} - b_{2p-1}), \qquad p = 1-M, 2-M, \ldots, N, \quad (8)$$

where $B = \int_K \mathrm{d}\kappa/\kappa_0$ denotes the relative bandwidth of K. Note that the left-hand side of (8) is evaluated in the narrow bandwidth approximation as $B \ll 1$, and that a more complicated expression holds when B is close to unity.

3 Plane-wave scattering by a homogeneous and isotropic sphere

In order to exemplify the theory in Sec. 2, consider the direct scattering problem of a plane-wave $\hat{\boldsymbol{e}}\mathrm{e}^{\mathrm{i}\kappa\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}$ of unit amplitude impinging in the $\hat{\boldsymbol{k}}$-direction on a homogenous and isotropic sphere of radius a. Here, $\boldsymbol{x}$ measures the position vector in units of a, and $\kappa = ka$, where k denotes the wave number in free space. The target is assumed to be non-magnetic with lossless material parameters modeled by the permittivity $\epsilon = \epsilon(\kappa)$ relative to the free space background. The scattered electric field $\boldsymbol{E}_\mathrm{s}$ in the far-field region then reads [6]

$$\boldsymbol{E}_\mathrm{s}(\kappa;\boldsymbol{x}) = \frac{\mathrm{e}^{\mathrm{i}\kappa x}}{x}\mathbf{S}(\kappa;\hat{\boldsymbol{x}})\cdot\hat{\boldsymbol{e}} + \mathcal{O}(x^{-2}) \text{ as } x\to\infty,$$

where $x = |\boldsymbol{x}|$, and the scattering dyadic $\mathbf{S}$ is the mapping from the incoming plane-wave (evaluated at the origin) into the amplitude of an outgoing spherical wave in the $\hat{\boldsymbol{x}}$-direction. This scattering problem is causal for $\hat{\boldsymbol{x}} = \hat{\boldsymbol{k}}$ in the sense that the scattered field in the forward direction cannot precede the incident field. As a consequence, the arguments in Refs. 2 and 3 suggest that $h(\kappa) = 4\hat{\boldsymbol{e}}^*\cdot\mathbf{S}(\kappa;\hat{\boldsymbol{k}})\cdot\hat{\boldsymbol{e}}/\kappa$ can be extended to a Herglotz function for a fixed electric polarization $\hat{\boldsymbol{e}}$. This Herglotz function has the property that $h(\kappa) = -h^*(-\kappa^*)$ for complex-valued κ [6].

Now, introduce $\sigma_\mathrm{ext}(\kappa) = \operatorname{Im} h(\kappa)$ as the extinction cross section (*i.e.*, the sum of the scattering and absorption cross sections) in units of the geometrical cross section area πa^2, *cf.*, the optical theorem in Ref. 6. From the T-matrix approach in Ref. 7, the low-frequency expansion of $h(\kappa)$ reads

$$h(\kappa) = 4\frac{\epsilon_\mathrm{s}-1}{\epsilon_\mathrm{s}+2}\kappa + \frac{4}{15}\frac{\epsilon_\mathrm{s}^4+25\epsilon_\mathrm{s}^3-15\epsilon_\mathrm{s}^2-49\epsilon_\mathrm{s}+38}{(\epsilon_\mathrm{s}+2)^2(2\epsilon_\mathrm{s}+3)}\kappa^3 + \mathcal{O}(\kappa^4) \text{ as } \kappa\to 0, \tag{9}$$

where $\epsilon_\mathrm{s} = \epsilon(0)$ denotes the static response of the target. The corresponding high-frequency expansion is assumed to satisfy $h(\kappa) = \mathcal{O}(1)$ as $\kappa\to\infty$ as suggested by the extinction paradox in Ref. 2. Thus, inserting $N = 2$ and $M = 0$ into (7), we conclude that the following two summation rules hold for the lossless dielectric sphere:[1]

$$\int_0^\infty \frac{\sigma_\mathrm{ext}(\kappa)}{\kappa^2}\,\mathrm{d}\kappa = 2\pi\frac{\epsilon_\mathrm{s}-1}{\epsilon_\mathrm{s}+2}, \qquad \int_0^\infty \frac{\sigma_\mathrm{ext}(\kappa)}{\kappa^4}\,\mathrm{d}\kappa = \frac{2\pi}{15}\frac{\epsilon_\mathrm{s}^4+25\epsilon_\mathrm{s}^3-15\epsilon_\mathrm{s}^2-49\epsilon_\mathrm{s}+38}{(\epsilon_\mathrm{s}+2)^2(2\epsilon_\mathrm{s}+3)}. \tag{10}$$

Here, the first integral is proportional to the diagonal elements of the polarizability dyadic, see Ref. 2. As expected, both integrals vanish as $\epsilon_\mathrm{s}\to 1$. Also note that the second integral becomes unbounded whereas the first integral approaches 2π in the high-contrast limit as $\epsilon_\mathrm{s}\to\infty$. Furthermore, since $-1/h(\kappa)$ defines a new Herglotz function, we have

$$-\frac{1}{h(\kappa)} = -\frac{1}{4}\frac{\epsilon_\mathrm{s}+2}{\epsilon_\mathrm{s}-1}\kappa^{-1} + \frac{1}{60}\frac{\epsilon_\mathrm{s}^2+27\epsilon_\mathrm{s}+38}{2\epsilon_\mathrm{s}+3}\kappa + \mathcal{O}(\kappa^2) \text{ as } \kappa\to 0,$$

with the additional assumption that $-1/h(\kappa) = \mathcal{O}(1)$ as $\kappa\to\infty$. Since $\operatorname{Im}(-1/h(\kappa)) = \operatorname{Im} h(\kappa)/|h(\kappa)|^2$ and $|h(\kappa)|^2 = 4\sigma_\mathrm{rcs}(\kappa;\hat{\boldsymbol{k}})/\kappa^2$, we obtain the following summation rule when $N = 1$ and $M = 0$ are inserted into (7):

$$\int_0^\infty \frac{\sigma_\mathrm{ext}(\kappa)}{\sigma_\mathrm{rcs}(\kappa;\hat{\boldsymbol{k}})}\,\mathrm{d}\kappa = \frac{\pi}{30}\frac{\epsilon_\mathrm{s}^2+27\epsilon_\mathrm{s}+38}{2\epsilon_\mathrm{s}+3}, \tag{11}$$

where $\sigma_\mathrm{rcs}(\kappa;\hat{\boldsymbol{k}})$ measures the bistatic radar cross in the forward direction in units of πa^2. Note that the right-hand side of (11) is unbounded as $\epsilon_\mathrm{s}\to\infty$. Furthermore, the limiting value of (11) is $11\pi/25$ as $\epsilon_\mathrm{s}\to 1$ which is rather intriguing since one would expect both the integrand and the integral to vanish when the scatterer approaches free space. The extinction cross section and the bistatic radar cross section in the forward direction are depicted in Fig. 2 for the lossless permittivities $\epsilon = 2$ and $\epsilon = 4$ independent of κ. A numerical integration of the two curves in Fig. 2 (except for the one labeled PEC) verifies that (10) and (11) hold with a relative error which can be made arbitrary small.

The corresponding low-frequency expansions of $h(\kappa)$ and $-1/h(\kappa)$ in the perfectly electric conducting (PEC) limit are

$$h(\kappa) = 2\kappa + \frac{113}{45}\kappa^3 + \mathcal{O}(\kappa^4) \text{ as } \kappa\to 0, \qquad -\frac{1}{h(\kappa)} = -\frac{1}{2}\kappa^{-1} + \frac{113}{180}\kappa + \mathcal{O}(\kappa^2) \text{ as } \kappa\to 0.$$

[1] The first integral in (10) with weighting factor $1/\kappa^2$ is also valid in the lossy case as discussed in Refs. 2 and 3.

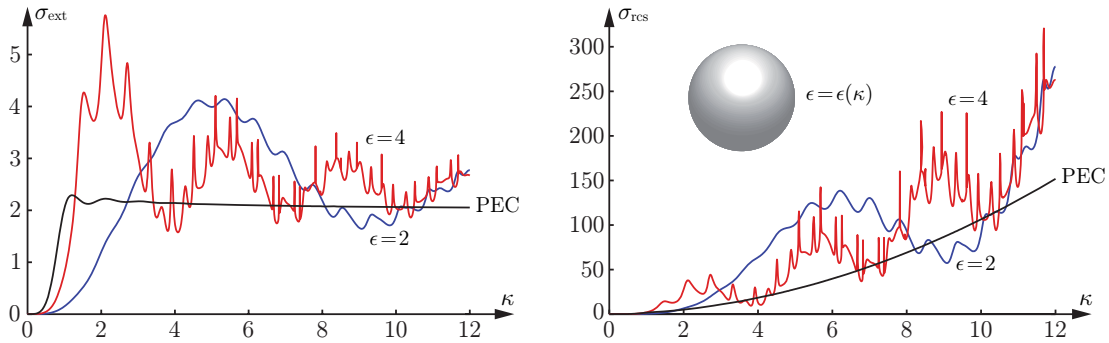

Figure 2: The extinction cross section (left figure) and the bistatic radar cross section in the forward direction (right figure) for a homogeneous and isotropic sphere of radius a. Note that both $\sigma_{\text{ext}}(\kappa)$ and $\sigma_{\text{rcs}}(\kappa;\hat{\boldsymbol{k}})$ are measured in units of the geometrical cross section area πa^2.

Thus, the three summation rules for the PEC sphere corresponding to (10) and (11) read

$$\int_0^\infty \frac{\sigma_{\text{ext}}(\kappa)}{\kappa^2}\,\mathrm{d}\kappa = \pi, \qquad \int_0^\infty \frac{\sigma_{\text{ext}}(\kappa)}{\kappa^4}\,\mathrm{d}\kappa = \frac{113\pi}{90}, \qquad \int_0^\infty \frac{\sigma_{\text{ext}}(\kappa)}{\sigma_{\text{rcs}}(\kappa;\hat{\boldsymbol{k}})}\,\mathrm{d}\kappa = \frac{113\pi}{90}. \tag{12}$$

Also the extinction cross section and the bistatic radar cross section in the forward direction are depicted in Fig. 2 for the PEC sphere. A numerical integration of the curves labeled PEC shows that also the summation rules in (12) hold to arbitrary precision. Due to the non-negative character of the integrands in (10) to (12), these summation rules can also be restated as physical bounds with respect to any finite frequency interval as suggested by (8).

4 Conclusions

It is concluded that (7) and (8) show great potential for deriving new physical bounds in electromagnetic theory. The results in this paper can also be generalized to include functions which are bounded in magnitude by unity, *e.g.*, transmission and reflection coefficients, by mapping the unit disk into the upper half plane using a linear fractional transformation. A variety of new summation rules can also be established by observing that the composition of two Herglotz functions again defines a new Herglotz function.

References

[1] C. Sohl. *Dispersion Relations for Extinction of Acoustic and Electromagnetic Waves*. Licentiate thesis, Lund University, Department of Electrical and Information Technology, P.O. Box 118, S-221 00 Lund, Sweden, 2007.

[2] C. Sohl, M. Gustafsson, and G. Kristensson. "Physical limitations on broadband scattering by heterogeneous obstacles". *J. Phys. A: Math. Theor.*, vol. 40, pp. 11165–11182, 2007.

[3] C. Sohl, M. Gustafsson, and G. Kristensson. "Physical limitations on metamaterials: Restrictions on scattering and absorption over a frequency interval". *J. Phys. D: Applied Phys.*, vol. 40, pp. 7146–7151, 2007.

[4] M. Gustafsson, C. Sohl, and G. Kristensson. "Physical limitations on antennas of arbitrary shape". *Proc. R. Soc. A*, vol. 463, pp. 2589–2607, 2007.

[5] H. M. Nussenzveig. *Causality and dispersion relations*. London: Academic Press, 1972.

[6] R. G. Newton. *Scattering Theory of Waves and Particles*. New York: Dover Publications, 2nd edn., 2002.

[7] V. V. Varadan, Y. Ma, V. K. Varadan, and A. Lakhtakia. "Scattering of waves by spheres and cylinders". In *Field Representations and Introduction to Scattering*, V. V. Varadan, A. Lakhtakia, and V. K. Varadan, eds., Amsterdam: Elsevier Science Publishers, vol. 1 of *Handbook on Acoustic, Electromagnetic and Elastic Wave Scattering*, chap. 4, pp. 211–324. 1991.