MATRIX-BASED THEORY AND PRACTICE OF RADIO-POLARIMETRIC SELF-CALIBRATION

Johan Hamaker
Netherlands Foundation for Research in Astronomy NFRA/ASTRON
P.O. Box 2, 7990 AA Dwingeloo, Netherlands
E-mail:hamaker@astron.nl

1 Theory

Radio interferometrists traditionally approach their trade from a scalar viewpoint that ignores the vector nature of electromagnetic radiation. Polarization effects are introduced later, as an afterthought so to say, and can only be treated in an approximate *quasi-scalar* way as a perturbation.

A proper theory must take vector signals as its starting point, which naturally calls for matrix algebra as the appropriate mathematical tool. A matrix-based theory of interferometry has been developed over the past decade in a series of papers that culminated in a theory of matrix self-calibration in [1]. I give a brief sketch here.

The radiation amplitude in the signal path of antenna j is represented by a vector $\boldsymbol{e}_j$ and its propagation by multiplications with a *Jones matrix*, $\boldsymbol{e}'_j = \boldsymbol{J}_j\, \boldsymbol{e}_j$. The coherency in an interferometer jk is represented by the coherency matrix $\boldsymbol{E}_{jk} = \boldsymbol{e}_j \boldsymbol{e}_k^\dagger$ and its propagation by the 'Measurement Equation'

$$\boldsymbol{E}'_{jk} = \boldsymbol{J}_j\, \boldsymbol{E}_{jk}\, \boldsymbol{J}_k^\dagger. \tag{1}$$

The coherency is the Fourier(-like) transform of the brightness distribution $\boldsymbol{B}(\boldsymbol{l})$ which is also a 2×2 matrix. The Stokes brightness I and *polvector* $\boldsymbol{p} = (Q, U, V)$ form an alternative repesentation of this matrix.

The obvious formal analogy with the scalar-interferometer description begs the question whether the powerful scalar method of *self-calibration* (affectionately known as *selfcal*) can also be generalised. [1] shows that it can indeed.

Analogous to the brightness/gain scale ambiguity in the scalar-selfcal solution, the matrix solution is subject to a matrix ambiguity. Given a set of observed coherencies $\boldsymbol{E}'_{jk}$, selfcal seeks to fit a set of Jones matrices $\{\boldsymbol{J}_j\}$ and a set of coherencies $\{\boldsymbol{E}_{jk}\}$ that fit (1) (in a LSQ sense to account for the presence of noise). [1] shows that the set of true values in (1) is one member of the family of solutions

$$\{\boldsymbol{J}'_j\} = \{\boldsymbol{J}_j\, \boldsymbol{X}^{-1}\}; \qquad \{\boldsymbol{E}'_{jk}\} = \{\boldsymbol{X}\, \boldsymbol{E}_{jk}\, \boldsymbol{X}^\dagger\} \quad \text{or} \quad \{\boldsymbol{E}'_{jk}\} = \mathcal{X}\, \{\boldsymbol{E}_{jk}\} \tag{2}$$

and argues that other solutions probably do not exist. Because of the presence of the unknown operator $\mathcal{X}$, I prefer to speak of self-*alignment*. To complete the calibration, we must eliminate $\boldsymbol{X}$ by a process of *post-calibration*.

The *poldistortion* transformation $\boldsymbol{E}' = \mathcal{X}\boldsymbol{E}$ is an in-place transformation: It redistributes power between the elements of $\boldsymbol{E}$ and hence between the Stokes brightnesses for each sky point but, like scalar selfcal, does not scatter it *between* points: It conserves dynamic range. The operator $\mathcal{X}$ can be factored as

$$\mathcal{X} = |x|^2\, \mathcal{Y}\, \mathcal{H} \tag{3}$$

where

- $|x|^2$ is the positive scale factor that is familiar from scalar selfcal.

- $\mathcal{Y}$ is a *polrotation* of $\boldsymbol{p}$ in QUV space; a familiar manifestation of it in quasi-scalar polarimetry is the effect of a phase difference between X and Y or R and L channels.
- $\mathcal{H}$ is represents a *polconversion* between Stokes I and $\boldsymbol{p}$; it includes not only the familiar 'leakage' from I to $\boldsymbol{p}$ but also its inverse. I shall refer I flux polconverted into $\boldsymbol{p}$ flux as *polconversion* $\boldsymbol{p}$ *flux*.

2 Processing a simulated observation

2.1 Sources, instrument and observation

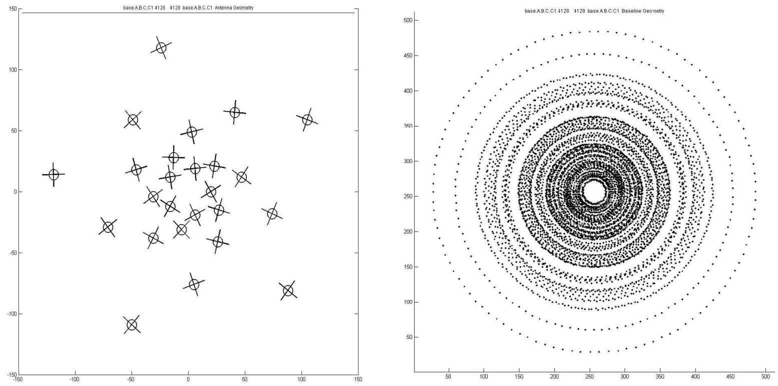

Figure 1: Synthesis-telecope geometry.

Left: Antenna positions and feed orientations.
Right: uv coverage.

To test the above theory in practice, I processed a simulated observation. To avoid the many distractions that the imperfections of real instruments and data offer, I simplified the problem through artificial but physically valid assumptions, to wit:

- Source field and antenna array in an Equatorial plane: No projection effects.
- Point sources only, on grid positions: Perfect CLEANing possible.
- Antennas shifted to grid positions for every integration interval: No *uv* convolution required.

There are 130 sources on a 256^2 grid. Flux and linear polarization distributions are representative of the sky in the lower GHz range; one source has weak circular polarization. The instrument has a LOFAR-like layout with 25 antennas on 5 spiral arms; hour-angle coverage in the *uv* plane is uniform. The feeds are linearly polarized in uniformly distributed random orientations; instrumental gain and phase have large errors for which only a coarse correction is a priori known. A wild Faraday rotation with peak-to-peak excursions of more than a full turn is included.

2.2 Self-alignment procedure

As indicated, self-aligning seeks to find a combination of a *source model* representing the observed sky and an *instrument model* representing the observing errors. The former is constructed by conventional CLEANing of sky images, the latter by fitting Jones matrices to the observed coherencies and those derived from the source model.

The self-align cycle sketched in Table 1 mirrors that for scalar self-alignment, but includes a number of polarization-specific additions. I note the following:

- In cycle 1 only Stokes I is included in the source model; this is enough to drastically reduce the gain and phase errors in the subsequent Jones-matrix fit, so that the $\boldsymbol{p}$ map in cycle 2 is good enough to also extract a $\boldsymbol{p}$ source model.

Cycles				Operation
1	2	3	4	⇒ *data*
x	x	x	x	⇒ *Raw observed coherencies*
x	x	x	x	Enter next cycle
				SELF-ALIGNMENT
x				Correct using best error estimates
x	x	x	x	Convert to image (FT) in Stokes format
x	x	x	x	⇒ *I and $\boldsymbol{p}$ dirty maps*
		x	(x)	**Polreversion**: Invert polconversion
		x	x	⇒ *'Leakage'-free dirty maps*
x	x	x	x	Source model from I map (CLEAN)
	x	x	x	Source model from $\boldsymbol{p}$ maps (polCLEAN)
x	x	x		Convert (FT) to **coherency in matrix format**
x	x	x		Estimate Jones matrices from obs. and model coherencies
x	x	x		Go to next cycle
				POST-CALIBRATION
			x	Minimise $\sum V^2$ in source model
			x	Interpret errors in terms of receiver hardware
			x	Compare stage with prior knowledge
			x	Correct coherency model for these errors
			x	Convert to final image in Stokes format

Table 1: Schematic representation of four self-alignment iterations followed by post-calibration.

- Polconversion adds to all sources in a sky map a polarization component with one and the same relative polarization $\boldsymbol{p}/I$. Since we know that most sources are 'un'polarized, polconversion is easy to recognise in a comparison of the I source model and the $\boldsymbol{p}$ maps, both for the human eye and a computer algorithm that *polreverts* the error. To distinguish true and polconversion $\boldsymbol{p}$ flux, one needs the $\boldsymbol{p}$ source model; therefore, polreversion cannot be applied before the third cycle.

  Although Paper IV describes it as a post-calibration operation, this polreversion should be applied as early as possible to prevent polconversion $\boldsymbol{p}$ flux from being incorporated in the source model.

- The true linear polvectors reside in the QU plane. Polrotation may rotate them in this plane but also tilt the plane as a whole, giving rise to V components for every linearly polarized source. Their appearance is a likely sign of an error, because significant real V values are rare. The tilt is easily correctible by a simple algorithm. After that truly circularly polarized sources will clearly stand out. Table 2 shows an example.

- I CLEANing is conventional. In $\boldsymbol{p}$ CLEANing it is assumed that $\boldsymbol{p}$ flux can only occur where there is I flux. The algorithm looks for the maximum of $|\boldsymbol{p}|$, then subtracts a component from each of the three $\boldsymbol{p}$ images.

- The Jones-matrix fit is a non-linear least-square problem that can be solved by a general-purpose or a specialised algorithm. In both cases, convergence is fast for gain, but polrotation requires a complementary algorithm.

- For self-alignment, only the overall Jones matrices $\boldsymbol{J}_j$ for the entire signal paths from source to correlator input are meaningful. In post-calibration, it may be relevant to factor them, e.g. into (Faraday and parallactic) rotation, feed error and electronic complex gain: $\boldsymbol{J}_j = \boldsymbol{G}_j \boldsymbol{D}_j \boldsymbol{R}_j$. $\boldsymbol{J}_j$ depends on 7 relevant real parameters. In addition to 1 rotation and 2 complex receiver gains, only 2 feed parameters can be determined, but the equations can be supplemented by a condition on the feed, e.g. that it be near-perfect, i.e. $||\boldsymbol{D} - \mathbf{I}||$ minimal.

I flux true	lin.-pol. flux true	V flux true	V flux self-al rotated	V flux $\sum V^2$ minimised
0.9660	0.1323	0.0000	-0.0054	-0.0002
1.0000	0.1200	0.0000	0.0051	0.0003
0.6071	0.0685	0.0000	-0.0029	-0.0002
0.9107	0.0608	0.0000	-0.0009	-0.0002
0.6221	0.0365	-0.0030	-0.0009	-0.0025
0.2008	0.0293	0.0000	0.0011	-0.0001
0.2718	0.0266	0.0000	-0.0012	-0.0001
0.6301	0.0196	0.0000	-0.0007	0.0000
0.2319	0.0178	0.0000	-0.0000	-0.0000
0.0436	0.0069	0.0000	0.0002	-0.0001
0.0402	0.0023	0.0000	-0.0000	-0.0000
0.0493	0.0007	0.0000	0.0000	0.0000
0.0027	0.0003	0.0000	0.0000	0.0000

Table 2: The effect of minimising $\sum V^2$.

Shown is a list of the polarized sources in the final model. The true V fluxes are compared with those obtained through self-alignment before and after the de-rotation. De-rotation reduces all of them to near-zero except for one source which is indeed the one that is truly polarized.

3 Results and conclusion

The simulation study produced all the results that were hoped for. The outcome after four cycles of self-alignment fully confirms the theory of [1]. Moreover, matrix self-alignment proved to be quite robust. Already after one cycle, in which it is mainly the receiver complex gains that are adjusted, the factoring of (3) becomes quite manifest in the results — but admittedly the simulated source field is a very 'easy' one.

The visible manifestations of polconversion and QU-to-V polrotation had not been anticipated, but they could hardly have failed to turn up and gratifyingly corroborate the theory. Their easy correction demonstrates the great virtue of being able to deal with in-place poldistortion transformations that apply to the entire image: The latter trait allows one to extract correcting parameters from the image as a whole instead of having to deal with individual sources in isolation (or even with interference) from the others.

Faraday self-calibration also proved to work: Matrix self-alignment lines u the $\boldsymbol{p}$ vectors scattered in different directions in successive time intervals, leaving only an unknown zero point of the rotation undefined. The quality of $\boldsymbol{p}$ maps will be much better than we have known it — but to ascertain the absolute orientation of linear polarization recourse must still be had to external data.

Although the methods described are not a panacaea for all polarimetric woes, they constitute a considerable step forward. The absence of software that properly incorporates them is a rather formidable obstacle to their adoption in practice. Although aips++ has endorsed the Measurement Equation as its basis, its matrix selfcal methods predate the theory of [1] and lean on obsolescent and inadequate concepts. The future seems to lie with new software packages that use large parts of the aips++ infrastructure, e.g. for the data processing of the LOFAR phased-array telescope.

Reference:

[1] J.P. Hamaker: "Understanding Radio Polarimetry, IV: "The full-coherency analogue of scalar self-calibration: Self-alignment, dynamic range and polarimetric fidelity". *Astron. and Astrophys. Suppl.*, **143**, 515-534, May I 2000.