A comparison of different techniques for the calculation of antenna coupling within a cavity

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ABSTRACT

We present and compare different techniques to efficiently solve coupled Hallen's and Pocklington's equations in order to calculate the mutual coupling between two dipole antenas that are located within a rectangular cavity. Our solution techniques involve the use of a computationally efficient Green's function, the approximation of the non-singular part of the Green's function by cubic splines, the regularization of Pocklington's equation, and an analytic approach for electrically small antennas.

INTRODUCTION

The concepts of antenna theory play an important part in Electromagnetic Compatibility (EMC) analysis if Electromagnetic Interference (EMI) sources and victims are modelled as unintentionally transmitting and receiving antennas, respectively [1, 2]. In this case the coupling between EMI sources and victims is determined from the mutual antenna impedance between two antennas. For two antennas 1 and 2 the mutual impedance $Z_{12} = Z_{21}$ is provided by a 2×2 impedance matrix which relates the input currents and input voltages of the antennas according to [3]

$$V_1 = Z_{11} I_1 + Z_{12} I_2 \,, \tag{1}$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2 \,. \tag{2}$$

Since EMI sources or victims often are enclosed by an environment which acts as a resonator we will in the following study antennas that are enclosed by a rectangular cavity. This poses difficulties in the analytic and numerical calculation of the mutual impedance since in this case the electromagnetic interaction is characterized by a dyadic Green's function which, a priori, mathematically is expressed by slowly converging series expansions [4, 5].

For the calculation of the mutual impedances it is advantageous to use integral formulas which require the calculation of antenna currents I_1 and I_2. Explicitly, we have for the calculation of Z_{12} the formula [3]

$$Z_{12} = -\frac{1}{I_1^{\mathrm{t}} I_2^{\mathrm{oc1}}} \int_{\text{antenna 2}} \boldsymbol{E}^{\mathrm{t21}}(\xi) \cdot \boldsymbol{J}_2^{\mathrm{oc1}}(\xi)\, d\xi \,. \tag{3}$$

Clearly, an analogous expression is valid for Z_{21}. The quantities in this expression refer to two different situations. The first is indexed by "t" and is given by a transmitting antenna 1 in absence of antenna 2. That is, I_1^{t} is the input current of a transmitting antenna in absence of antenna 2 and $\boldsymbol{E}^{\mathrm{t21}}(\xi)$ is the electric field at the position of antenna 2 which is generated by antenna 1 if antenna 2 is removed from its position. The second situation is indexed by "oc1" and is given by a transmitting antenna 2 in the presence of antenna 1 which is open-circuited. The corresponding quantities I_2^{oc1} and $\boldsymbol{J}_2^{\mathrm{oc1}}(\xi)$ refer to the input current and current distribution, respectively, of antenna 2 in this situation.

We will now discuss four different techniques to calculate the currents that are required to evaluate (3) within a rectangular cavity:

1. Numerical solution of coupled Hallén's equations
2. Numerical solution of coupled Pocklington's equations
3. Numerical solution of regularized Pocklington's equations
4. Analytical solution of coupled Pocklington's equations by an electrically small antenna approximation

The electric field $\boldsymbol{E}^{\mathrm{t21}}(\xi)$ that enters (3) will be obtained from the integrated current $I_1^{\mathrm{t}}(\xi)$, weighted with the electric Green's function. In the following we will work in the frequency domain with time-dependency $\exp(j\omega t)$ and wavenumber $\beta = \omega/c$.

COUPLED HALLÉN'S AND POCKLINGTON'S EQUATIONS

Hallén's equation and Pocklington's equation are standard integral equations for the calculation of the electric current $\boldsymbol{J}$ on a linear antenna [6, 7]. They are derived from the constraint that the tangential component of the magnetic vector potential $\boldsymbol{A}$ and the electric field $\boldsymbol{E}$, respectively, vanishes on a perfectly conducting antenna surface. We split the total fields in a scattered and an incident part and formally write Hallén's and Pocklington's equation as

$$\left[\int \bar{\bar{\mathbf{G}}}^{A}(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')d\boldsymbol{r}' + \boldsymbol{A}^{\mathrm{inc}}(\boldsymbol{r})\right]_{\tan} = 0\,, \qquad \left[\int \bar{\bar{\mathbf{G}}}^{E}(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{J}(\boldsymbol{r}')d\boldsymbol{r}' + \boldsymbol{E}^{\mathrm{inc}}(\boldsymbol{r})\right]_{\tan} = 0\,, \tag{4}$$

with the dyadic Green's function $\bar{\bar{\mathbf{G}}}^{A}(\boldsymbol{r},\boldsymbol{r}')$ for the magnetic vector potential and $\bar{\bar{\mathbf{G}}}^{E}(\boldsymbol{r},\boldsymbol{r}')$ the dyadic electric Green's function. For N coupled antennas the conditions (4) must be enforced on each antenna surface such that in this case they constitute N coupled integral equations. More explicitly, we denote by $\boldsymbol{r}^{(i)}$, $\boldsymbol{r}^{(j)}$ a position vector that points to the surface of antenna i, j, respectively, with $i, j = 1, \ldots, N$. If we further assume that the antennas are aligned with one rectangular coordinate axis, say, the z-axis, and adopt a thin-wire approximation the coupled integral equations (4) acquire the form

$$\sum_{j=1}^{N} \int_{\text{antenna}j} G^{A}_{zz}(\boldsymbol{r}^{(i)},\boldsymbol{r}^{(j)}) I(\boldsymbol{r}^{(j)})\, d\boldsymbol{r}^{(j)} = -A^{\mathrm{inc}}_{z}(\boldsymbol{r}^{(i)})\,, \quad \sum_{j=1}^{N} \int_{\text{antenna}j} G^{E}_{zz}(\boldsymbol{r}^{(i)},\boldsymbol{r}^{(j)}) I(\boldsymbol{r}^{(j)})\, d\boldsymbol{r}^{(j)} = -E^{\mathrm{inc}}_{z}(\boldsymbol{r}^{(i)})\,, \tag{5}$$

with $I(\boldsymbol{r}^{(i)})$ the N unknown antenna currents. In order to apply the methods of moments to (5) we expand each current as a linear combination of basis functions, $I(\boldsymbol{r}^{(i)}) = \sum_k \alpha^{(i)}_k I_k(\boldsymbol{r}^{(i)})$.

In order to solve the Hallén's integral equation system we divide each antenna into $2M-1$ intervals of length $h^{(j)} = L^{(j)}/(2M-1)$ with $L^{(j)}$ the length of antenna j, choose the midpoints of the intervals as matching points $\boldsymbol{r}^{(i)}_k$, and further take pulse functions as basis functions, $I_k = P_k$, with $P_k(\boldsymbol{r}^{(j)}) = 1$ if $z^{(j)} \in \left[z^{(j)}_k - \frac{h^{(j)}}{2}, z^{(j)}_k + \frac{h^{(j)}}{2}\right]$ and $P_k(\boldsymbol{r}^{(j)}) = 0$ otherwise. Then we obtain the algebraic system of equations

$$\sum_{j=1}^{N} \sum_{k=1}^{2M-1} \alpha^{(j)}_k Z^{(j)A}_k(\boldsymbol{r}^{(i)}_l) = -A^{\mathrm{inc}}_z(\boldsymbol{r}^{(i)}_l)\,, \qquad i = 1,\ldots,N\,, \quad l = 1,\ldots,2M-1 \tag{6}$$

which provides $N \times (2M-1)$ equations for $N \times (2M-1)$ unknowns $\alpha^{(j)}_k$. The matrix elements $Z^{(j)A}_k(\boldsymbol{r}^{(i)}_l)$ are given by

$$Z^{(j)A}_k(\boldsymbol{r}^{(i)}_l) = \int_{z^{(j)}_k - \frac{h^{(j)}}{2}}^{z^{(j)}_k + \frac{h^{(j)}}{2}} G^{A}_{zz}(\boldsymbol{r}^{(i)}_l, \boldsymbol{r}^{(j)})\, d\boldsymbol{r}^{(j)} \approx G^{A}_{zz}(\boldsymbol{r}^{(i)}_l, \boldsymbol{r}^{(j)}_k) |\boldsymbol{r}^{(i)}_l - \boldsymbol{r}^{(j)}_k| \int_{z^{(j)}_k - \frac{h^{(j)}}{2}}^{z^{(j)}_k + \frac{h^{(j)}}{2}} \frac{1}{|\boldsymbol{r}^{(i)}_l - \boldsymbol{r}^{(j)}|} d\boldsymbol{r}^{(j)}\,. \tag{7}$$

In the last line we isolated the Coulomb singularity of the Green's function, the remaining integral is calculated analytically. Due to the thin-wire approximation the distance $|\boldsymbol{r}^{(i)}_l - \boldsymbol{r}^{(j)}|$ is bounded from below by the antenna radius.

In order to solve the Pocklington's integral equation system we divide each antenna into $2M$ intervals of length $h^{(j)} = L^{(j)}/(2M)$, choose the $2M-1$ points between adjacent intervals as matching points $\boldsymbol{r}^{(i)}_k$, and further take piecewise sinusoidal functions as basis functions, $I_k = S_k$, with $S_k(\boldsymbol{r}^{(j)}) = \frac{\sin\beta\left(z^{(j)} - z^{(j)}_{k-1}\right)}{\sin\beta h^{(j)}}$ if $z^{(j)}_{k-1} \le z^{(j)} \le z^{(j)}_k$, $S_k(\boldsymbol{r}^{(j)}) = \frac{\sin\beta\left(z^{(j)}_{k+1} - z^{(j)}\right)}{\sin\beta h^{(j)}}$ if $z^{(j)}_k \le z^{(j)} \le z^{(j)}_{k+1}$, and $S_k(\boldsymbol{r}^{(j)}) = 0$ otherwise. Then we obtain the algebraic system of equations

$$\sum_{j=1}^{N} \sum_{k=1}^{2M-1} \alpha^{(j)}_k Z^{(j)E}_k(\boldsymbol{r}^{(i)}_l) = -E^{\mathrm{inc}}_z(\boldsymbol{r}^{(i)}_l)\,, \qquad i = 1,\ldots,N\,, \quad l = 1,\ldots,2M-1 \tag{8}$$

which provides $N \times (2M-1)$ equations for $N \times (2M-1)$ unknowns $\alpha^{(j)}_k$. The matrix elements $Z^{(j)E}_k(\boldsymbol{r}^{(i)}_l)$ turn out to be

$$Z^{(j)E}_k(\boldsymbol{r}^{(i)}_l) = -\frac{j\omega}{\beta \sin\beta h^{(j)}} \left[G^{A}_{zz}(\boldsymbol{r}^{(i)}_l, \boldsymbol{r}^{(j)}_{k+1}) + G^{A}_{zz}(\boldsymbol{r}^{(i)}_l, \boldsymbol{r}^{(j)}_{k-1}) - 2\cos\beta h^{(j)} G^{A}_{zz}(\boldsymbol{r}^{(i)}_l, \boldsymbol{r}^{(j)}_k)\right]. \tag{9}$$

One should note that for the calculation of the expressions (7) and (9) for the matrix elements $Z^{(j)A}_k(\boldsymbol{r}^{(i)}_l)$ and $Z^{(j)E}_k(\boldsymbol{r}^{(i)}_l)$, respectively, the Green's function G^{A}_{zz} neither needs to be integrated nor differentiated. This is a considerable advantage which reduces the main computational effort to the computation of the Green's function for the various positions of pairwise intervals of the discretized antennas.

Since we assume that the antennas are enclosed by a lossy, rectangular cavity we have to use a corresponding Green's function. We will take advantage of a specific Ewald representation for the Green's function [8, 9, 10] which is suitable to calculate precise values both in the source region and at resonance. In our case we have to adapt the representation to complex wavenumbers in order to include cavity losses [11]. To further speed up calculation time the smooth part of the Green's function is approximated by cubic splines. For a fine discretization it is then not necessary to calculate all required values from the Ewald representation. The Ewald representation is only used to provide the sample values for the interpolation.

REGULARIZATION OF POCKLINGTON'S EQUATION

For the solution of the coupled equations (6) and (8) we first have to choose the length $h^{(j)}$ for the discretization of each antenna. Smaller values are expected to yield more accurate values but require more computation time. It turns out that in terms of the number of intervals the solution of Hallén's equation converges much faster if compared to the solution of Pocklington's equation. This circumstance is known from the analogous situation in free space and due to the stronger Coulomb singularity of the electric Green's function if compared to the Coulomb singularity of the Green's function of the magnetic vector potential. Condition numbers of the Hallén's system usually are of the order of 10^1 while for the Pocklington's system they are of the order 10^4, indicating less stable solutions.

This situation can be improved by a regularization [12] of Pocklington's equation. Pocklington's equation is of the form $\hat{G}I = -E^{\text{inc}}$ and we may split the linear integral operator $\hat{G}$ in a free space part $\hat{G}_{\text{free}}$ and a remaining part $\hat{G}_{\text{mode}}$. Then the unknown current I can be determined from

$$(1 + \hat{G}_{\text{free}}^{-1}\hat{G}_{\text{mode}})I = I_{\text{free}} \tag{10}$$

with $I_{\text{free}} := -\hat{G}_{\text{free}}E^{\text{inc}}$ the free space solution of the antenna problem. Within the method of moments the operators $\hat{G}_{\text{free}}^{-1}$ and $\hat{G}_{\text{mode}}$ turn to matrices and the system (10) becomes considerably more stable than the original Pocklington's equation. It is characterized by condition numbers of the order of 10^1. Clearly, this is due to the removal of the Coulomb singularity which is represented in the free space solution I_{free}. In practice, it is computationally efficient to obtain I_{free} for a fine discretization and then to use a more coarse discretization to solve (10) for the current I.

SMALL ANTENNA APROXIMATION

We consider an antenna as electrically small if, for a given excitation, the resulting antenna current can be approximated by one sinusoidal basis function. In this case the Pocklington's equation only contains one unknown per antenna current which is the unknown factor in front of a single basis function. Then the algebraic system of linear equations that is provided by the method of moments reduces such that an analytical solution is feasible. For two antennas the corresponding solution can be put into the expression (3) of the mutual impedance. For two electrically small antennas that are aligned to the z-axis this yields the approximate but simple solution

$$Z_{12} = -\frac{j\omega L_2}{2\beta\sin(\beta L_2/2)}\Big[G_{zz}^A\big(\boldsymbol{r}_2, \boldsymbol{r}_1 + (L_1/2)\hat{z}\big) + G_{zz}^A\big(\boldsymbol{r}_2, \boldsymbol{r}_1 - (L_1/2)\hat{z}\big) - 2\cos(\beta L_2/2)G_{zz}^A\big(\boldsymbol{r}_1, \boldsymbol{r}_2\big)\Big] \tag{11}$$

with $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ the positions of the centers of the antennas and L_1, L_2 the lengths of the antennas, respectively. The unit vector in the z-direction is denoted by $\hat{z}$.

EXAMPLE

To illustrate our results we consider a cavity of dimensions $l_x = 6$m, $l_y = 7$m, and $l_z = 3$m. This geometry is similar to the one of a mode-stirred chamber at the University of Magdeburg. Ohmic losses of this chamber are approximately described by the frequency-dependent quality factor $Q(f) = 0.1\sqrt{f/\text{Hz}}$ [13]. We first consider two antennas of length $L_1 = L_2 =$ 2m and radius $\rho = 1$mm. They are aligned to the z-axis and their centers are positioned at $\boldsymbol{r}_1 = (l_x/2, l_y/2, l_z/2)$ and $\boldsymbol{r}_2 = (l_x/4, l_y/2, l_z/2)$. The first cavity resonance occurs at 32.9 MHz and represents the mode $\nu = n_x n_y n_z = 110$ with respect to the eigenfunctions $\varphi_\nu(\boldsymbol{r}) := \sin(k_x x)\sin(k_y y)\cos(k_z z)$ with $k_{x_i} = n_{x_i}\pi/l_{x_i}$. Up to 100 MHz there are a total of 19 cavity resonances. A simultaneous mode coupling of the two antennas to the same mode occurs for the modes 110, 130, 310, and 330.

The solutions of the coupled Hallén's and Pocklington's equations for this setup allow to compute the mutual impedance Z_{12}. The result is displayed in the left part of Fig. 1. As expected, a strong mutual coupling is observed if both antennas couple to the same mode. Minor resonace peaks occur if only one antenna couples to a specific mode. While the solutions of the Hallén's and Pocklington's equation agree well one should note that the required computational effort to obtain each

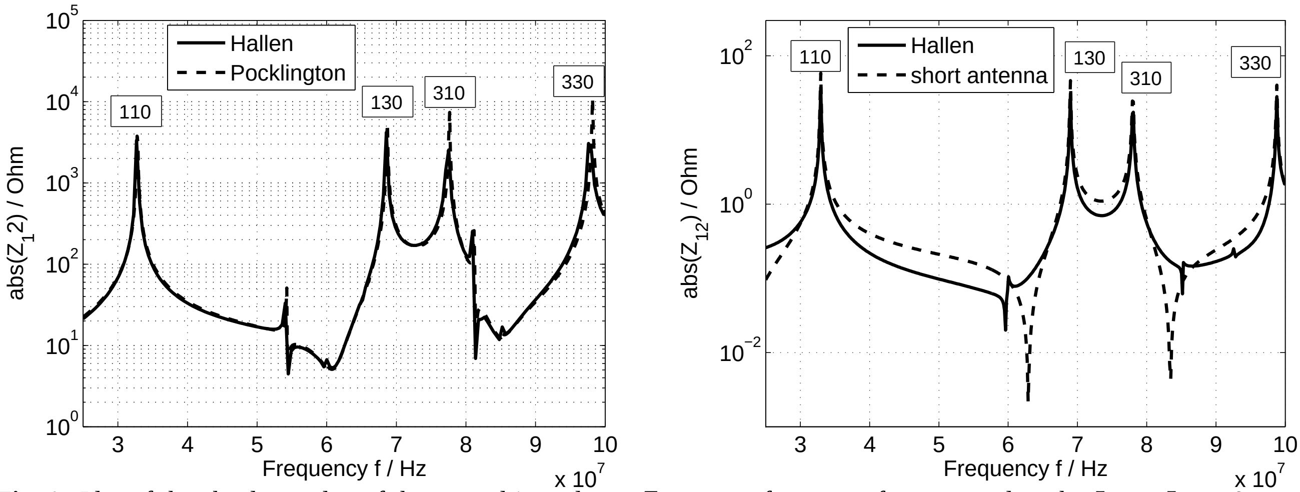

Fig. 1: Plot of the absolute value of the mutual impedance Z_{12} versus frequency for antenna lengths $L_1 = L_2 = 2\text{m}$ (left) and $L_1 = L_2 = 0.2\text{m}$ (right).

solution is different. In general, there is a trade-off between accuracy and computation time. As already indicated above, the solution of the regularized Pocklington's equation is prefered over the direct solution of Pocklington's equation. The Hallén's equation appears to be the even better choice. However, it must be remarked that Hallén's equation can only be applied if the electromagnetic excitation of a given problem can be expressed in terms of the incident magnetic vector potential $\boldsymbol{A}^{\text{inc}}$ which enters (6).

If the antennas are shortened by a factor of 10 to yield the lengths $L_1 = L_2 = 0.2\text{m}$ it is meaningful to use the approximate analytic formula (11) and compare it to a full method of moment solution. The corresponding result is shown in the right part of Fig. 1. It is seen that the approximate formula reproduces the dominant resonance peaks that characterize the simultaenous coupling to a specific mode. However, inbetween the dominant resonance peaks the mutual coupling is characterized by secondary resonances and cavity effects which are not properly taken into account by the approximate formula.

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