Time-Domain Analysis of Two-Dimensional Scattering Problems by Use of a Source-Model Technique

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1. INTRODUCTION

Time-domain integral equation based solution techniques have gradually shifted over the last couple of years into the focus of computational electromagnetics research [1]. In this paper, we present a mesh-free time-domain integral equation solution for the problem of electromagnetic scattering by a two-dimensional metallic cylinder illuminated by a TM (transverse magnetic) plane wave pulse. In the proposed solution, we adapt the frequency-domain source model technique (SMT) [2], which has been found to be efficient and versatile computational tool for analysis of time-harmonic wave scattering problems, to allow direct time-domain analysis of transient scattering problems. In this approach, instead of formulating the problem in terms of the current induced on the cylinder surface, it is formulated in terms of a fictitious current residing on a mathematical surface interior to the cylinder surface. A spatio-temporal division for attaining an explicit discretization scheme, which allows the use of a simple marching-on-in-time (MOT) algorithm, is presented. The use of an implicit discretization scheme is also discussed, and the advantages of such a scheme are outlined. Finally, the use of a combined-source formulation and its effect on the resulting stability is studied.

2. FORMULATION

Consider the scattering problem of a two-dimensional z-directed arbitrarily shaped cylinder made of a perfect electric conductor (PEC), whose surface is denoted by S, lying in a homogeneous isotropic medium of permittivity ε and permeability μ, and illuminated by a transient TM$_z$ incident wave plane with electric field $\mathbf{E}^{\text{inc}}(\mathbf{r},t)$. An SMT formulation of this problem is obtained by replacing the scatterer with an unknown fictitious surface current distribution $\mathbf{J}_{\mathbf{S_i}}$, uniform in the z-direction, residing on a mathematical cylindrical surface S_i enclosed inside S, and radiating in an unbounded space with the same electric parameters as those of the ambient medium. The SMT formulation in the time-domain is obtained from the boundary condition of zero total tangential electric field on the cylinder surface, which yields the following continuous two-dimensional integral equation

$$\frac{\eta}{2\pi}\int\int_{S_{i,2D}}\int_{-\infty}^{t-P/c}\frac{1}{\sqrt{c^2(t-t')^2-P^2}}\cdot\frac{\partial\mathbf{J}_{S_i}\left(\boldsymbol{\rho}',t'\right)}{\partial t'}\,dt'ds'=\mathbf{E}^{\text{inc}}\left(\boldsymbol{\rho},t\right),\quad \boldsymbol{\rho}\in S_{2D}. \tag{1}$$

Here, c is the speed of light in the ambient medium, $\eta=\sqrt{\mu/\varepsilon}$ is the impedance of the medium, $S_{i,2D}$ is the circumference of the two-dimensional cross-section of the mathematical surface S_i, S_{2D} is the circumference of the cross-section of the cylinder, $\boldsymbol{\rho}$ and $\boldsymbol{\rho}$' are two-dimensional vectors on the xy plane and $P=|\boldsymbol{\rho}-\boldsymbol{\rho}'|$. Next, the fictitious current distribution $\mathbf{J}_{S_i}$ is approximated by a discrete set of N z-directed electric current filaments positioned at discrete locations $\boldsymbol{\rho}_n$ on the surface S_i (see Fig. 1). By introducing a simple time discretization, the time derivative of the equivalent current surface density can be written as

$$\frac{\partial\mathbf{J}_s\left(\boldsymbol{\rho}',t'\right)}{\partial t'}=\hat{\boldsymbol{z}}\sum_n\sum_{k'}\mathcal{I}_{n,k'}\delta_s\left(\boldsymbol{\rho}'-\boldsymbol{\rho}_n\right)T_{n,k'}\left(t'\right), \tag{2}$$

where $\mathcal{I}_{n,k'}$ denotes the unknown temporal derivative of the current amplitude of the nth fictitious source at the k'th retarded time step, δ_s is the surface delta function, $T_k\left(t'\right)$ is the set of pulse basis functions given by

$$T_{n,k'}(t')=\begin{cases}1 & k'\Delta t<t'+\dfrac{P_{n,\min}}{c}<(k'+1)\Delta t\\ 0 & \text{otherwise}\end{cases}, \tag{3}$$

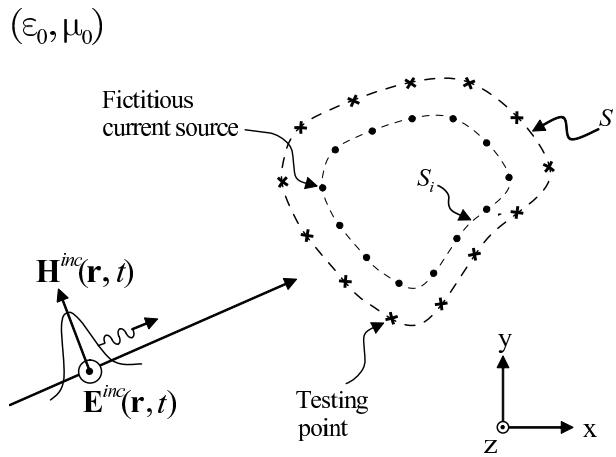

Figure 1: Simulated equivalence for the exterior region created by the fictitious sources and sampled at the testing points on the cylinder's surface.

and $P_{n,\min}$ is the minimal distance between the nth source and any of the testing points on the surface of the cylinder. Substituting (2) into (1) and imposing the latter at the time intervals $t_i = (i+1)\Delta t$ over the set of M testing points $\boldsymbol{\rho}_m$ on the circumference of the cross-section of the cylinder, the continuous integral equation is reduced to its discretized form written in matrix form as

$$[\mathbf{Z}_{\text{last}}]\,\boldsymbol{\mathcal{I}}_i = \mathbf{V}_i - \sum_{k'=0}^{i-1} \left[\mathbf{Z}_{\text{prev}(i-k')}\right]\boldsymbol{\mathcal{I}}_{k'}, \tag{4}$$

where

$$[\mathbf{Z}_{\text{last}}]_{m,n} = \left[1 - U\left(\frac{P_{m,n} - P_{\tilde{m}_n,n}}{c\Delta t} - 1\right)\right] \text{acosh}\left(\frac{c\Delta t}{P_{m,n}}\left[\left\lfloor\frac{P_{m,n} - P_{\tilde{m}_n,n}}{c\Delta t}\right\rfloor + \frac{P_{\tilde{m}_n,n}}{c\Delta t} + 1\right]\right), \tag{5}$$

$$\boldsymbol{\mathcal{I}}_j = \begin{bmatrix} \mathcal{I}_{1,j} \\ \mathcal{I}_{2,j} \\ \vdots \\ \mathcal{I}_{N,j} \end{bmatrix}, \qquad \mathbf{V}_j = -\frac{2\pi c}{\eta}\begin{bmatrix} E_z^{\text{inc}}(\boldsymbol{\rho}_1, (j+1)\Delta t) \\ E_z^{\text{inc}}(\boldsymbol{\rho}_2, (j+1)\Delta t) \\ \vdots \\ E_z^{\text{inc}}(\boldsymbol{\rho}_M, (j+1)\Delta t) \end{bmatrix}, \tag{6}$$

and

$$\left[\mathbf{Z}_{\text{prev}(i-k')}\right]_{m,n} = \begin{cases} \text{acosh}\left(\frac{c\Delta t}{P_{m,n}}\left[(i-k') + 1 + \frac{P_{\tilde{m}_n,n}}{c\Delta t}\right]\right) - \text{acosh}\left(\frac{c\Delta t}{P_{m,n}}\left[(i-k') + \frac{P_{\tilde{m}_n,n}}{c\Delta t}\right]\right), & i\text{-}k' < \left\lfloor\frac{P_{m,n} - P_{\tilde{m}_n,n}}{c\Delta t}\right\rfloor\text{-}1 \\ U\left(\frac{P_{m,n} - P_{\tilde{m}_n,n}}{c\Delta t}\text{-}1\right)\text{acosh}\left(\frac{c\Delta t}{P_{m,n}}\left[\left\lfloor\frac{P_{m,n} - P_{\tilde{m}_n,n}}{c\Delta t}\right\rfloor + \frac{P_{\tilde{m}_n,n}}{c\Delta t} + 1\right]\right), & i\text{-}k' = \left\lfloor\frac{P_{m,n} - P_{\tilde{m}_n,n}}{c\Delta t}\right\rfloor\text{-}1 \\ 0, & \text{otherwise} \end{cases}. \tag{7}$$

Here, U is the Heaviside step function given by

$$U(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}, \tag{8}$$

and

$$P_{m,n} = |\boldsymbol{\rho}_n - \boldsymbol{\rho}_m| \tag{9}$$

is the distance between the nth source point and the mth testing point. Also, $\lfloor x \rfloor$ is an operator that gives the largest integer less than or equal to x, and $\tilde{m}_n$ is defined as the index of the testing point nearest to

the nth source, so that $P_{\tilde{m}_n,n} = \left|\boldsymbol{\rho}_n - \boldsymbol{\rho}_{\tilde{m}_n}\right|$.

The generally formulated matrix system in (4) can represent either an explicit or implicit discretization schemes. To attain an explicit discretization scheme, two requirements must be fulfilled. First, the scatterer geometry should allow the positioning of the fictitious sources so that the source nearest to a given testing point should be closer to that testing point than to all the other testing points. The second requirement is that the time intervals should be small enough to allow only the nearest source to influence a given testing point during a given time interval. Denoting by $\tilde{n}_m$ the source nearest to the mth testing point, the requirements given above can be written as

$$\forall m : \tilde{m}_{\tilde{n}_m} = m, \tag{10}$$

and

$$\forall m, n \neq \tilde{n}_m : \frac{P_{m,n} - P_{m,\tilde{n}_m}}{c\Delta t} \geq 1. \tag{11}$$

The aforementioned requirements of the explicit discretization scheme ensure that the matrix $[\mathbf{Z}_{\text{last}}]$ is a diagonal matrix or a permutation of a diagonal matrix. This enables the use of a marching on in time (MOT) algorithm whereby the transient scattering problem can be solved without necessitating any matrix inversion. In an implicit discretization scheme, the time intervals are large enough to allow more than one source to influence each testing point during a given time interval. In this case, the solution is naturally more computationally intensive. However, the error due to late-time instabilities in this case is often markedly smaller than that encountered in solutions based on explicit discretization scheme.

Finally, the Time-domain SMT scheme given above is extended to the case of combined source formulation. In this formulation, we use a combination of electric source and magnetic sources sharing the same yet to be determined amplitude. The electric source is no other than the z-directed electric current filament used in the previous formulation, while the magnetic source is a z-directed filament comprising a continuum of infinitesimal dipoles of magnetic current whose directions are tangential to $S_{i,2D}$. The combined-source formulation is obtained along the same lines as given above for the electric source formulation and the resulting expressions are similar except for a more complicated expression for the matrices $[\mathbf{Z}_{\text{last}}]_{m,n}$ and $\left[\mathbf{Z}_{\text{prev}(i-k')}\right]_{m,n}$.

3. RESULTS

To compare the results obtained by the use of the time-domain source-model technique with the results obtained by a MOT method solution of the magnetic field integral equation (MFIE) [3], we consider a circular PEC cylinder illuminated by a TM$_z$ incident plane wave with electric field amplitude given by

$$\mathbf{E}^{\text{inc}}(\boldsymbol{\rho}, t) = \frac{2}{\sqrt{\pi}\eta\tau} \exp\left(-\frac{4\left(t - \boldsymbol{\rho}\cdot\hat{\mathbf{x}}/c\right)^2}{\tau^2}\right)\hat{\mathbf{z}}. \tag{12}$$

Here, $\tau = a/c$ is the time required for a wave to travel one cylinder radius, which makes the pulse width approximately equal to the diameter of the cylinder $2a$. In Fig. 2, the normalized backscattered far-field obtained by the use of both an electric-source and combined-source formulation of the SMT, is shown and compared with numerical results obtained by the MOT MFIE solution [3]. The time $t' = 0$ corresponds to the time that the peak of the incident pulse would reach an observer at a distance $\rho_0 = 100a$ if the incident pulse were reflected from the center of the cylinder. The numerical parameters for the source-model technique are as follows: 30 filamentary current sources were placed on a cylindrical surface of a radius $r_0 = 0.75a$ and 120 testing points were specified on the cylinder surface. We used the implicit solution scheme of the time-domain source-model technique. This was because the explicit scheme was sensitive to the source surface location and required that the sources be placed very close to the cylinder surface for attaining a stable solution. It can be clearly seen that all the results are in good agreement before the late-time instability emerges. Comparing the results of the electric source formulation and the combined source formulation, the improvement in stability when using combined sources is evident.

References

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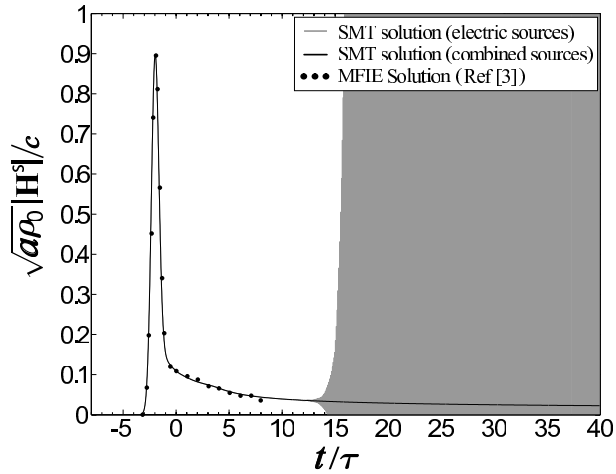

Figure 2: Normalized backscattered far-field of circular cylinder with radius a, obtained by the electric-source formulation of the SMT (bright solid line), by the combined-source formulation of the SMT (dark solid line), and by a MOT solution of the MFIE [3] (dots).

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