WINDOWED RADON TRANSFORM FRAMES AND PULSED BEAM EXPANSIONS FOR RADIATION FROM TIME-DEPENDENT APERTURE SOURCES

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ABSTRACT

WRT frames are a new class of frames in the Hilbert space of band-limited functions in $\mathbb{R}^3$, whose elements are obtained by shifting and rotating a window function $\psi(\boldsymbol{x})$, $\boldsymbol{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$. As such, WRT are structured upon a 3D×2D phase-space lattice of points and directions. We discuss the frame properties: the phase-space lattice, the overcompleteness, and the expansion formula, wherein a given function is expanded in terms of its local spectral directions. We then apply the WRT expansion to derive pulsed beam summation representations for time-dependent radiation problems.

INTRODUCTION AND MOTIVATION

Time-dependent radiation from distributed sources can be analyzed directly in space-time domain via Greens functions or, alternatively, in the spectral domain of time-dependent plane-waves. Both representations lead to distributed integrals, although the actual field may be localized in one of these domains or in both using stationary point (constructive interference) evaluation that leads to ray-type representations. Asymptotic failure (e.g., caustics) in one domain can be avoided by switching to the other domain. This spatial-spectral interplay is best described in the phase space, a mixed domain comprising of both spatial and spectral coordinates, where the ray localization takes place on the Lagrange sub-manifold (the geometrical optics skeleton).

The discussion above suggests the use of beam propagators[1] as basis functions that tend to localize the radiation process a-priori, leading to localized uniform representations that are structured about the geometrical optics skeleton. Beam fields are good candidates since: (a) they can be matched locally to the source distribution thus localizing the radiation integrals; (b) further localization is due to the fact that only those beams that pass near a given observation point actually contribute there; and (c) beams are tracked locally along the axis and they do not fail at caustics or foci.

Beam summation representations for *time harmonic* fields have been used in various applications (see recent reviews in [1]–[2]). Mathematically, they are based on a windowed Fourier transform (WFT) expansion of the spatial source distribution that expresses the source in a phase space comprising its local spectrum about the window center. The WFT extracts the local radiation properties of the source thus enhancing the beam basis functions that conform with these properties.

The continuous spectrum of beams is highly overcomplete, hence it may be sampled and expressed as a discrete set. At first, such representations were based on the (critically complete) Gabor series, but more recently it has been shown that overcomplete representations using WFT frames are preferable [3]. The formulations above have dealt, however, only with monochromatic fields, hence when applied to ultra wideband (UWB) fields, the beam axes are changed with frequency and the propagators have to be recalculated for each frequency. A *key step* toward the formulation of an UWB theory has been the introduction in [2] of a particular frequency-scaling of the WFT frame, that has led to a comprehensive UWB beam summation representation. It has the following required properties: (i) the beam lattice (the set of the beam trajectories) is *frequency independent*; (ii) the expansion coefficients are *local* (depend on the local properties of the source) and *stable* for all frequencies (this is achieved if one uses isodiffracting Gaussian beam (ID-GB) windows); and (iii) the ID-GB propagators have a convenient form that can be tracked through the medium, and are described by frequency independent propagation parameters that need to be calculated only once for all the frequencies.

The formulation in [2] is phrased in the frequency domain, and it is convenient if the sources and the field or the system parameters are given/required in a frequency-by-frequency format. If the sources are given as a function of time, it might be preferable/required to calculate the field and the system parameters in the time domain.

[1] we used the generic term "beams" for both time-harmonic and pulsed-beam fields.

In view of features (i)–(iii), the theory may be reformulated in the time domain so that all the operations are performed directly on the space-time data, and the propagators are pulsed beams (PB). Such formulation is our motivation here.

Returning to the time domain, we note that the plane-wave representation is extracted from the space-time source distribution via the so called slant stack transform (SST) which is, in fact, a Radon transform in space-time [4]–[6]. The time-domain beam summation representation is based, therefore, on a windowed version of the SST that expresses the local space-time spectrum about the window center (by analogy with the frequency domain formulations discussed above where the beam expansions are structured upon a WFT of the source distribution). The windowed SST has been introduced originally in [7] with application in [8]. So far, however, there was only a continuous phase-space version of this transform, leading to field representations as a continuous spectrum of pulsed beam (PB) propagators that emerge from all space-time points and directions in the source domain. Clearly, this representation is highly overcomplete and practically inconvenient.

This issue is addressed in the present work where we introduce a new class of frames, termed the windowed Radon transform (WRT) frames, which, when applied in wave theory, yields a discrete version of the windowed SST. We start by formulating the WRT frames in a general 3D domain $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. We consider a 3D formulation since our interest is in wave theory, but the WRT can also be applied in other disciplines (e.g., image representation) where a 2D formulation is required. The theory is then applied to the analysis of time-dependent aperture source distribution $u_0(x_1, x_2, t)$ in the $z = 0$, radiating into the half space $z > 0$. In the new formulation the field is described by a discrete set of PB propagators that emerge from a set of space-time points and directions in the source domain (see Fig. 1). Their excitation coefficients are obtained via the WRT processing of the space-time data u_0 in the 3D domain (x_1, x_2, ct), where c is the wavespeed (Figs. 2 and 3).

WINDOWED RADON TRANSFORMED FRAMES: GENERAL PROPERTIES

The windowed Radon transform (WRT) frame is a new class of frames in $\mathbb{H}_2^\Omega$, the Hilbert space of band-limited, square-summable functions in $\mathbb{R}^3$. The frame elements are obtained by shifting and shearing a proper window function $\psi(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, and are given by (see Fig. 1 with $x_3 \to ct$)

$$\psi_{\boldsymbol{m},\boldsymbol{n}}(\boldsymbol{x}) = \psi[x_1 - m_1\bar{x}_1,\ x_2 - m_2\bar{x}_2,\ x_3 - m_3\bar{x}_3 - n_1\bar{\xi}_1(x_1 - m_1\bar{x}_1) - n_2\bar{\xi}_2(x_2 - m_2\bar{x}_2)]\,. \tag{1}$$

$\psi_{\boldsymbol{m},\boldsymbol{n}}(\boldsymbol{x})$ are structured upon a 5D phase space, comprising a discrete lattice of points $\boldsymbol{x}_{\boldsymbol{m}} = (m_1\bar{x}_1, m_2\bar{x}_2, m_3\bar{x}_3)$ tagged by the index $\boldsymbol{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$ with $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ denoting the lattice's unit cell; and a discrete lattice of directions $\boldsymbol{\xi}_{\boldsymbol{n}} = (n_1\bar{\xi}_1, n_2\bar{\xi}_2)$, indexed by $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$ with $(\bar{\xi}_1, \bar{\xi}_2)$ being the unit cell of the shearing operation about a given preferred axis, which is taken here to be x_3. The unit cell dimensions are determined by the data bandwidth Ω. A given function $f(\boldsymbol{x}) \in \mathbb{H}_2^\Omega$ is thereby described as a discrete sum of shifted and tilted windows $\psi_{\boldsymbol{m},\boldsymbol{n}}$

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{m},\boldsymbol{n}} a_{\boldsymbol{m},\boldsymbol{n}}\psi_{\boldsymbol{m},\boldsymbol{n}}(\boldsymbol{x}), \qquad a_{\boldsymbol{m},\boldsymbol{n}} = \langle f(\boldsymbol{x}), \varphi_{\boldsymbol{m},\boldsymbol{n}}(\boldsymbol{x})\rangle. \tag{2}$$

The expansion coefficients $a_{\boldsymbol{m},\boldsymbol{n}}$ are the projections of $f(\boldsymbol{x})$ onto the dual frame $\varphi_{\boldsymbol{m},\boldsymbol{n}}(\boldsymbol{x})$ which has the same form as $\psi_{\boldsymbol{m},\boldsymbol{n}}(\boldsymbol{x})$ except that it involves a "dual window" $\varphi(\boldsymbol{x})$. This operation is recognized as a localized generalization of the conventional Radon transform (Fig. 1), hence the term WRT frames. The reconstruction of f is the discrete inverse transform. Explicit expressions for $\psi_{\boldsymbol{m},\boldsymbol{n}}$ and $\varphi_{\boldsymbol{m},\boldsymbol{n}}$ are given for the class of isodiffracting (ID) windows, that are also matched to the $(\boldsymbol{x}_{\boldsymbol{m}}, \boldsymbol{\xi}_{\boldsymbol{n}})$ lattice to yield snug frames. We also construct the dual frame set and determine the conditions for these frames to be snug.

LOCAL SST AND PULSED BEAM SUMMATION REPRESENTATION

The WRT frame with the ID windows is applied next to the problem of radiation into the half space $z > 0$ due to a time-dependent source distribution $u_0(x_1, x_2, t)$ in the $z = 0$ plane (Fig. 1). We use the WRT frame formulation to analyze the data u_0 in the 3D domain (x_1, x_2, ct), c is the wavespeed, by projecting it on the dual frame $\varphi_{\boldsymbol{m},\boldsymbol{n}}(x_1, x_2, ct)$ i.e. (Fig. 2)

$$u_0(x_1, x_2, ct) = \sum_{\boldsymbol{m},\boldsymbol{n}} a_{\boldsymbol{m},\boldsymbol{n}}\psi_{\boldsymbol{m},\boldsymbol{n}}(x_1, x_2, ct), \qquad a_{\boldsymbol{m},\boldsymbol{n}} = \langle u_0(x_1, x_2, ct), \varphi_{\boldsymbol{m},\boldsymbol{n}}(x_1, x_2, ct)\rangle. \tag{3}$$

where $\psi_{\boldsymbol{m},\boldsymbol{n}}(x_1,x_2,ct) = \psi[x_1 - m_1\bar{x}_1,\ x_2 - m_2\bar{x}_2,\ ct - p c\bar{t} - n_1\bar{\xi}_1(x_1 - m_1\bar{x}_1) - n_2\bar{\xi}_2(x_2 - m_2\bar{x}_2)]$ and here $\boldsymbol{m} = (m_1, m_2, p) \in \mathbb{Z}^3$ with $(\bar{x}_1, \bar{x}_2, c\bar{t})$ denoting the lattice's unit cell (to avoid confusion we use different symbols for the time coordinates). Each frame element gives rise to an isodiffracting pulsed beam (ID-PB) propagator $B_{\boldsymbol{m},\boldsymbol{n}}(\boldsymbol{r},t)$ where $\boldsymbol{r} = (x_1, x_2, z)$ (Fig. 3), thus expressing the radiating field as a sum of PBs

$$u(\boldsymbol{r},t) = \sum_{\boldsymbol{m},\boldsymbol{n}} a_{\boldsymbol{m},\boldsymbol{n}} B_{\boldsymbol{m},\boldsymbol{n}}(\boldsymbol{r},t), \tag{4}$$

emerging from a discrete set of space-time points $(m_1\bar{x}_1, m_2\bar{x}_2, p\bar{t})$ and directions $(n_1\bar{\xi}_1, n_2\bar{\xi}_2)$ in the $z=0$ plane (Fig. 1).

The formulation is demonstrated for the radiation of pulsed focused aperture distribution.

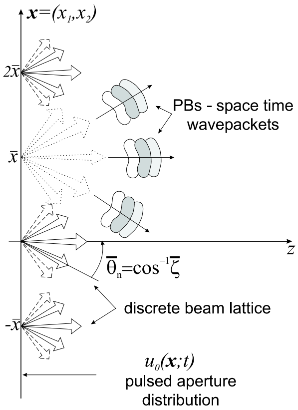

Figure 1

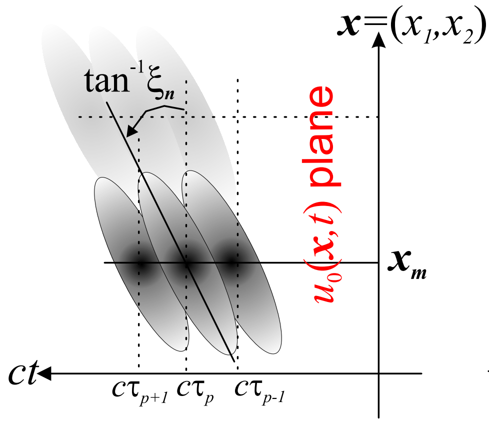

Figure 2

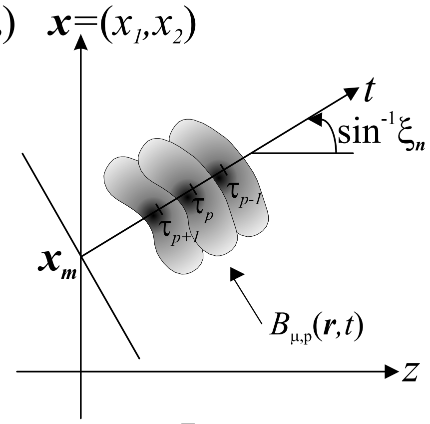

Figure 3

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