ANALYTICAL REGULARISATION TECHNIQUES IN SCATTERING AND PROPAGATION

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ABSTRACT

Analytical methods are important in identifying significant mechanisms in electromagnetic scattering, and in providing reliable benchmark solutions for the validation of purely numerical methods. Whether differential or integral equation formulations are used, standard numerical approaches often encounter difficulties of accuracy and convergence with scattering structures that feature cavities and sharp edges, and are of moderate or large size (in wavelengths). Analytical-numerical methods such as the Method of Regularisation provide solutions that are particularly useful when the more usual approaches such as the Method of Moments are ill-conditioned and computationally inaccurate. Some recent developments in these analytic and analytical-numerical methods are discussed.

INTRODUCTION

The impetus for effective methods in scattering and propagation of electromagnetic waves has many sources, such as communications, remote sensing and imaging; and accurate modelling of the wave interaction with its environment, an antenna or sensor, or a remotely sensed target, is crucial in assessing the feasibility of intended applications. Modern computing power makes feasible various numerical approaches hitherto impossible. If an integral equation with a Green's function kernel is used to formulate the scattering problem (such as the Electric Field Integral Equation(EFIE)), the Method of Moments, first popularised by Harrington [1], produces a finite system of linear algebraic equations for the coefficients of the basis functions that have been selected to represent the desired surface current. Alternatively direct discretisation of Maxwell's equations leads to methods such as the finite difference time domain (FDTD) method or the finite element method (FEM). The greatest success of both integral and differential approaches has been in the low to intermediate (or resonance) wavelength regime (where the scatterer dimensions are one to several wavelengths in size), and the computational cost limits the size of scattering problem that may be effectively solved. It should also be mentioned that high frequency methods such as physical optics or the geometrical theory of diffraction also have limitations, albeit different.

However the accuracy of such numerical methods particularly as the wavelength dimensions increase must be carefully assessed - a dense system of linear equations of hundreds of thousands of variables cannot avoid the possibility or indeed likelihood of ill-conditioning that substantially degrades the accuracy of any numerical solution whether it be computed by direct or iterative means (even preconditioning may not succeed). The mathematical foundations for error estimation in numerical solutions of the integral equations of electromagnetics has been carefully discussed in [2], and whilst reliable bounds for smooth closed scatterers (e.g. a sphere) have been obtained, open surfaces (with edges) and non-smooth surfaces are rather more problematic. Thus *analytical* methods and solutions are crucial to electromagnetics in providing reliable benchmark solutions to assess the accuracy of numerical methods in every aspect (surface discretisation, basis functions, etc.).

Another enduring reason for the study of analytical solutions to Maxwell's equations is the identification of dominant scattering mechanisms - the solution of canonical problems chosen to highlight certain features, such as edges or cavity-backed apertures, provide us with reliable quantitative predictions for target features to be more carefully modelled by purely numerical general purpose codes for arbitrarily shaped scatterers. A third reason is the development of more accurate "semi-analytical" or "analytical - numerical" methods applicable to a class of scatterers wider than the idealised canonical shapes. The method of regularisation (MoR) described in [3] falls into this class: it solves the difficulty of error estimation encountered with the integral equations of electromagnetics by analytically transforming them to well-conditioned second-kind matrix systems that have a firm basis for error estimation in Fredholm theory. Validation of general purpose computer scattering codes for complex objects incorporating edges and cavity-backed apertures, which themselves may enclose a variety of other scatterers, depends entirely upon comparison with the results of other proven approaches, analytical, computational or experimental. This paper surveys some recent developments in analytical regularisation methods that illustrate the continuing importance of analytical or analytical - numerical methods in electromagnetics.

THE METHOD OF REGULARISATION

The classic text [4] provides an excellent survey of known scattering results for a variety of mainly closed canonical surfaces (such as the sphere). The analytical regularisation methods described in [3] provide a powerful approach to some canonical open surfaces or cavity backed aperture models. They may be simply illustrated in the context of spherically-shaped cavities. Consider the open spherical shell of radius a with a circular aperture subtending an angle θ_0 at the origin. Locate an electric dipole on the z-axis a distance d ($< a$) in the positive direction from the origin; the z-axis is the axis of symmetry and the dipole moment of strength p is aligned with the axis. Let k denote the wavenumber.

A correct formulation of the diffraction problem providing a unique solution to Maxwell's equations is as follows. First the solution must be continuous across surface $r = a$, and satisfy the mixed boundary conditions

$$H_\phi\left(a-0,\theta\right) = H_\phi\left(a+0,\theta\right), 0 < \theta < \theta_0 \tag{1}$$

$$E_\theta\left(a-0,\theta\right) = E_\theta\left(a+0,\theta\right) = 0, \theta_0 < \theta < \pi. \tag{2}$$

Second, the scattered field must represent an outgoing spherical wave at infinity, and finally, the energy of the scattered field ($\vec{E}^s, \vec{H}^s$) in any arbitrary finite region V of space, including the cavity edges, must be finite:

$$\frac{1}{2}\int\int\int_V \left\{\epsilon_0\left|\vec{E}^s\right|^2 + \mu_0\left|\vec{H}^s\right|^2\right\} dV < \infty. \tag{3}$$

We seek the scattered field in the form

$$H_\phi^s = \frac{pk^2}{r}\sum_{n=1}^{\infty}\frac{2n+1}{n\left(n+1\right)}x_n P_n^1\left(\cos\theta\right) R_n(kr), \tag{4}$$

where $R_n(kr)$ equals $\zeta_n'\left(ka\right)\psi_n\left(kr\right)$ or $\psi_n'\left(ka\right)\zeta_n\left(kr\right)$ according as $r < a$ or $r > a$; here $\psi_n(x) = xj_n(x), \zeta_n(x) = xh_n^{(1)}(x)$ where $j_n, h_n^{(1)}$ denote the spherical Bessel functions. The finite energy condition (3) effectively constrains the unknown coefficient sequence x_n, yet to be found, to be square summable.

The internal and external fields are matched across the aperture by (1), (2), to produce the dual series equations

$$\sum_{n=1}^{\infty}\frac{2n+1}{n\left(n+1\right)}x_n P_n^1\left(\cos\theta\right) = 0, 0 < \theta < \theta_0, \tag{5}$$

$$\sum_{n=1}^{\infty}\left(1-\varepsilon_n\right)x_n P_n^1\left(\cos\theta\right) = \sum_{n=1}^{\infty}\alpha_n P_n^1\left(\cos\theta\right), \tag{6}$$

when $\theta_0 < \theta < \pi$; the coefficients α_n are known, and $\varepsilon_n = 1 + ika\psi_n'\left(ka\right)\zeta_n'\left(ka\right)/n(n+1)$ is an asymptotically small parameter as $n \to \infty$.

This system of dual series equations is equivalent to a first kind Fredholm integral equation (essentially the EFIE): it exhibits ill-conditioning that becomes especially noticeable under any discretisation process in which the cavity grid is increasingly refined. The method of regularisation (MOR) converts this system into a second-kind Fredholm matrix equation. The main tool exploited is the Abel integral transform applied to harmonic expansions comprising trigonometric and other special functions of hypergeometric type. The transformed equations take the form

$$\left(1-\varepsilon_m\right)x_m + \sum_{n=1}^{\infty}x_n\varepsilon_n R_{nm}^{(1)} = \alpha_m - \sum_{n=1}^{\infty}\alpha_n R_{nm}^{(1)}, \quad (m = 1, 2, \ldots), \tag{7}$$

$$R_{nm}^{(1)} = R_{nm}^{(1)}\left(\theta_0\right) = R_{nm}\left(\theta_0\right) - \frac{R_{n0}\left(\theta_0\right)}{1 - R_{00}\left(\theta_0\right)}R_{0m}\left(\theta_0\right),$$

and the "incomplete scalar product" $R_{nm}\left(\theta_0\right)$ equals

$$\frac{1}{\pi}\left\{\frac{\sin\left(n-m\right)\theta_0}{n-m} - \frac{\sin\left(n+m+1\right)\theta_0}{n+m+1}\right\}\ (n \neq m), \quad \text{or} \quad \frac{1}{\pi}\left\{\theta_0 - \frac{\sin\left(2n+1\right)\theta_0}{2n+1}\right\}\ (n = m). \tag{8}$$

The transformed equations are well-conditioned and standard numerical techniques based upon simple truncation methods are easily applicable and produce reliable results; moreover the computed solution of the system truncated to N_{tr} equations converges to the true solution as N_{tr} increases. In contradistinction to numerical methods (such as MOM) applied to the electrical field integral equation the convergence is guaranteed and can be reliably estimated on the basis of Fredholm theory. Thus scattered field results may be computed accurately.

This technique has been well developed for cavity structures possessing some spherical or rotational symmetry [3]: this includes spherical and spheroidal cavities with one or several apertures; the cavity may have metal or dielectric inclusions or may have an impedance lining [5]. Varying the geometric parameters of such canonical structures provides a large number of physically interesting benchmark solutions, against which the accuracy of solutions computed by another general purpose scattering code may be examined. The key feature of this approach is the analytical conversion of the more conventional formulations of scattering (differential or integral) to a second-kind matrix system for which there is a sound base of theory concerning the reliability of numerical solution methods.

MORE GENERAL SCATTERERS

The MoR has been extended to provide well-conditioned systems of equations governing the scattering from certain classes of non-canonical objects of more general and arbitrary shape. Although presently restricted to two-dimensional scattering, the extension has practical implications for calculations, such as radar cross-sections of aircraft engine inlets, or application to aspects of reflector antenna design [6]. Let L denote the cross-section of the scatterer; it is regarded as part of a smooth closed contour $S = \{p(\theta) = (x(\theta), y(\theta)) : \theta \in [-\pi, \pi]\}$, in which an aperture L' has been opened, and L corresponds to the subinterval $[-\theta_0, \theta_0]$. The single layer potential representation of the scattered field at a point q in terms of the surface current J_z is

$$E_z^s(q) = ikZ_0 \int_L G_2(k|p-q|)J_z(p)dl_p. \tag{9}$$

Here dl_p is the differential of arc length at $p \in L$, Z_0 denotes free space impedance, the free space Green's function $G_2(kR)$ equals $-\frac{i}{4}H_0^{(1)}(kR)$, and $R = |p-q|$. The boundary condition $E_z^s(p) = -E_z^{inc}(p)$ (the incident field) yields the usual (EFIE) for the unknown current density. Once this is found, all relevant physical quantities such as surface current, near field and far field radiation characteristics are easily calculated.

Introduce the new unknown function $z(\tau)$ defined by $z(\tau) = ikZ_0 J_z(p(\tau))\frac{dl}{d\tau}$ when $\tau \in [-\theta_0, \theta_0]$; outside this interval z is defined to be zero. The EFIE becomes

$$\int_{-\pi}^{\pi} G_2(kR(\theta,\tau))z(\tau)d\tau = -E_z^{inc}(p(\theta)),\ \theta \in [-\theta_0, \theta_0] \tag{10}$$

where $R(\theta,\tau) = R((x(\theta), y(\theta)), (x(\tau), y(\tau)))$. Equation (10) together with the vanishing requirement on z is completely equivalent to the EFIE.

Following the technique of [3], split the kernel of integral equation (10) into singular and regular parts,

$$H_0^{(1)}(kR(\theta,\tau)) = \frac{2i}{\pi}\ln\left|2\sin\frac{\theta-\tau}{2}\right| + H(\theta,\tau). \tag{11}$$

Expand H in a double Fourier series, and the incident field and the solution in Fourier series,

$$H(\theta,\tau) = \sum_{p=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h_{np}e^{i(n\theta+p\tau)}, \quad 2E_z^{inc}(p(\theta)) = \sum_{n=-\infty}^{\infty} g_n e^{in\theta}, \quad z(\tau) = \sum_{n=-\infty}^{\infty} \varsigma_n e^{in\tau}, \tag{12}$$

where $\theta, \tau \in [-\pi, \pi]$. We thus obtain dual series equations with exponential kernel,

$$\sum_{n=-\infty}^{\infty}{}' |n|^{-1}\varsigma_n e^{in\theta} - 2\sum_{n=-\infty}^{\infty} e^{in\theta}\sum_{p=-\infty}^{\infty} h_{n,-p}\varsigma_p = \sum_{n=-\infty}^{\infty} g_n e^{in\theta}, \quad \theta \in [-\theta_0, \theta_0], \tag{13}$$

$$\sum_{n=-\infty}^{\infty} \varsigma_n e^{in\theta} = 0, \qquad \theta \notin [-\theta_0, \theta_0]. \tag{14}$$

for which the unknowns $\{\varsigma_n\}_{n=-\infty}^{\infty}$ are to be found. (The prime signifies omission of the term $n = 0$.) An analytical process of regularisation, based on Abel's integral equation technique (see [3]), transforms the dual series equations to an infinite system of linear algebraic equations of the form

$$Z_m + \sum_{n=1}^{\infty} Z_n H_{nm} = C_m, \quad m = 1, 2, 3, \ldots \tag{15}$$

This regularised system is a Fredholm matrix equation of the second kind. Its mathematical properties ensure that truncation to a finite system produces a set of linear equations whose solution is guaranteed to converge to the exact solution as the truncation order N_{tr} increases.

PROPAGATION STUDIES WITH MoR

This approach may be adapted to study structures such as shielded microstrip structures, for example, a structure of rectangular cross-section in which perfectly conducting walls enclose the microstrip line lying on a dielectric substrate of thickness t. The method is effective particularly in determining the propagation constant of higher order modes and for limiting cases that are notoriously difficult to calculate with any accuracy [7].

Using standard modal expansions in the regions above and below the stripline and matching across the boundaries of these regions leads to some coupled dual series equations for the unknown modal coefficients; these are enforced on the line $y = t$ in the metallised region ($|x| < w/2$) and outside ($w/2 < |x| < L_x/2$). In this form the system is quite ill-conditioned for standard numerical solution based on discretising the intervals $|x| < w/2$, $w/2 < |x| < L_x/2$. Although it is more complex, the MoR may be applied to transform this system to a well-conditioned system that is suitable for the accurate determination of propagation constants.

CONCLUSION

By their nature, analytical regularisation methods are rather more difficult to apply than the conventional techniques of electromagnetics. However they possess the unique advantage of providing reliable and robust solutions to scattering problems that are of guaranteed accuracy, and thus provide benchmarks for the validation of general purpose codes. Other regularisation approaches exist in the literature, such as the Riemann-Hilbert method for two-dimensional scattering problems. The approach has been substantially developed by Shestopalov and co-workers; a survey of some recent work is [8]. An interesting recent example is the study of electromagnetic wave diffraction by a grating with a chiral layer [9]: this structure has the capability of transforming elliptically polarised waves to linear polarisation.

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