SCATTERING-MATRIX ANALYSIS OF 1D, 2D, AND 3D ARRAYS OF ACOUSTIC MONOPOLES AND ELECTROMAGNETIC DIPOLES[1]

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INTRODUCTION

During the past few years we have developed a systematic methodology for determining the propagation constants of traveling waves on periodic arrays of electrically (or acoustically) small radiators and scatterers. The methodology, which is founded upon a spherical-wave source scattering-matrix description of the array elements, employs general reciprocity and power conservation relations to greatly reduce the number of unknowns in the scattering matrices. For array elements that radiate and scatter electric and magnetic dipoles only, Poisson summation formulas and Floquet mode analyses combine to produce readily solvable transcendental equations for the propagation constants versus frequency (k–β diagrams) of the traveling waves on one-, two-, and three-dimensional periodic arrays. This paper outlines the steps in the methodology and shows the k–β diagrams that result from its application to periodic arrays of penetrable spheres.

ELECTRIC DIPOLE ANTENNAS

For an electrically small, lossless, z-directed electric-dipole antenna, the spherical-wave source scattering-matrix equations for linear, single-port, time-harmonic ($e^{-i\omega t}$) antennas reduces to [1]

$$b_0 = \Gamma a_0 + R E_z^0 \,, \qquad b = T a_0 + S E_z^0 \tag{1}$$

where a_0 and b_0 are the amplitudes of the ingoing and outgoing propagating mode, respectively, in the feed line of the dipole antenna, b is the coefficient of the dipole field radiated and scattered by the dipole antenna (in the far field, $\mathbf{E} = -b[e^{ikr}/(kr)]\sin\theta\hat{\boldsymbol{\theta}}$, (r,θ,ϕ) being the usual spherical coordinates and $k = \omega/c$ with c the speed of light), E_z^0 is the z component of the electric field incident upon the phase center of the electric dipole, and the reflection, receiving, transmitting, and scattering coefficients of the dipole antenna are denoted by Γ, R, T, and S, respectively. Writing T as $|T|e^{i\psi_T}$, reciprocity and power conservation show that the transmitting, receiving, and scattering coefficients can be expressed merely in terms of the reflection coefficient $\Gamma = |\Gamma|e^{i\psi_\Gamma}$ and the phase ψ_T of the transmitting coefficient; specifically

$$T = \frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\frac{\eta_0 k^2}{Y_0 \pi}}\,\sqrt{1-|\Gamma|^2}\,e^{i\psi_T}\,, \quad R = \sqrt{\frac{3}{2}}i\sqrt{\frac{\pi Y_0}{\eta_0 k^2}}\,\sqrt{1-|\Gamma|^2}\,e^{i\psi_T}\,, \quad S = \frac{3}{4}i\left(-|\Gamma|\,e^{i(2\psi_T-\psi_\Gamma)}+1\right) \tag{2}$$

in which Y_0 is the admittance of free space and η_0 is the characteristic admittance of the dipole antenna's feed line. With the passive lossless dipole antennas terminated in a lossless load with reflection coefficient given by $\Gamma_L = e^{i\psi_L}$, b is given by

$$b = S_e E_z^0 \tag{3}$$

where $S_e = |S_e|e^{i\psi_e} = TR/(e^{-i\psi_L} - \Gamma) + S$ satisfies the relationship $|S_e| = (3/2)\sin\psi_e$.

For an infinite 1D periodic array of electric dipoles perpendicular to the array axis, separated by a distance d, and terminated in a lossless load, the following transcendental equation for the propagation constants β of the traveling waves supported by this array results from equating the field incident on any one dipole to the sum of the fields scattered by all the other dipoles [1]

$$\begin{aligned}\frac{2}{3}(kd)^3\cos\psi_e + \big\{(kd)^2\ln\left[2(\cos kd - \cos\beta d)\right] + (kd)\left[F(kd+\beta d) - F(\beta d - kd)\right] \\ + \left[G(kd+\beta d) + G(\beta d - kd)\right]\big\}\sin\psi_e = 0, \quad kd < \beta d.\end{aligned} \tag{4}$$

In deriving (4), we have used the summation formulas

$$\sum_{j=1}^{\infty}\frac{\cos na}{n} = \frac{1}{2}\ln\frac{1}{2(1-\cos a)} = -\ln\left[2\sin(a/2)\right], \qquad \sum_{j=1}^{\infty}\frac{\sin na}{n} = \frac{\pi - a}{2}, \quad 0 < a < 2\pi \tag{5a}$$

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$$\sum_{j=1}^{\infty} \frac{\cos na}{n^2} = \frac{\pi^2}{6} - \frac{\pi a}{2} + \frac{a^2}{4}, \qquad \sum_{j=1}^{\infty} \frac{\sin na}{n^3} = \frac{\pi^2 a}{6} - \frac{\pi a^2}{4} + \frac{a^3}{12}, \quad 0 < a < 2\pi \tag{5b}$$

and the approximations

$$\sum_{j=1}^{\infty} \frac{\sin na}{n^2} \equiv F(a) \approx -0.1381 \sin a + 0.03212 \sin 2a - 0.9653 a \ln(a/\pi), \quad 0 < a < \pi \tag{6a}$$

with $F(a) = -F(2\pi - a)$ for $\pi \leq a < 2\pi$ and

$$\sum_{j=1}^{\infty} \frac{\cos na}{n^3} \equiv G(a) \approx 1.3328 - 0.1424 \cos a + 0.01094 \cos 2a + 0.4902 a^2 \ln(a/\pi) - 0.2417 a^2, \quad 0 < a < \pi \tag{6b}$$

with $G(a) = G(2\pi - a)$ for $\pi \leq a < 2\pi$. The equation (4) gives an implicit expression for the normalized propagation constants βd of the traveling waves supported by the 1D periodic array of lossless, passive, electrically small dipoles perpendicular to the array axis as a function of the normalized spacing kd and the phase ψ_e of the effective scattering coefficient. Although a closed-form expression cannot be found for βd as a function of kd and ψ_e as was possible for a 1D periodic array of lossless, passive, acoustically small isotropic radiators [2], the implicit expression (4) can readily be solved numerically for βd given kd and ψ_e.

For an infinite 1D periodic array of electric dipoles parallel to the array axis, separated by a distance d, and terminated in a lossless load, a transcendental equation for β can similarly be found as [1], [3]

$$-\frac{2}{3}(kd)^3 \cos\psi_e + \{kd\,[F(kd + \beta d) - F(\beta d - kd)] + [G(kd + \beta d) + G(\beta d - kd)]\} \sin\psi_e = 0, \quad kd < \beta d. \tag{7}$$

Figure 1 shows the kd–βd diagrams obtained by numerically solving (4) (transverse excitation) and (7) (longitudinal excitation) for an array of electrically small gold nanospheres (with radius a equal to 1/3 the separation distance d) that behave approximately as a lossless plasma with relative dielectric constant obeying the Drude model at optical frequencies [4], [5]. The scattering coefficient S_e for the nanospheres equals $-i(3/2)b_1^{sc}$, where b_1^{sc} is the electric-dipole Mie scattering coefficient [6]. Since our theory is restricted to unattenuated waves, only curves in the slow-wave region ($\beta d > kd$) are shown in Figure 1. (It can be proven [2] that unattenuated fast waves ($\beta d < kd$) are not supported by 1D periodic arrays of electrically small scatterers.) Note that the curve for transverse excitation is asymptotic to the $\beta d = kd$ (light-cone) line, while the curve for longitudinal excitation intersects the $\beta d = kd$ line. The localization of the curves to the region of kd between approximately 0.88 and 1.01 corresponds to the region close to the first resonance of the Mie electric-dipole scattering coefficient at $ka = 0.313$. (The magnitude of the Mie magnetic-dipole coefficient and all other multipole coefficients are negligible at these frequencies compared to the magnitude of the Mie electric-dipole coefficient.) The corresponding curves obtained by Park and Stroud [5] from a static field approximation are shown in red in Figure 1.

COUPLED ELECTRIC AND MAGNETIC DIPOLES

For orthogonal electric and magnetic dipole scattering with the electric dipole moment in the x direction and the magnetic dipole moment in the y direction, the spherical-wave source scattering-matrix along with the reciprocity and power conservation relations lead to [7]

$$b_e = S_e E_x^0, \qquad b_m = S_m H_y^0 / Y_0 \tag{8}$$

with scattering coefficients $S_e = |S_e| e^{i\psi_e}$ and $S_m = |S_m| e^{i\psi_m}$ obeying $|S_e| = (3/2)\sin\psi_e$ and $|S_m| = (3/2)\sin\psi_m$, where b_e and b_m are the coefficients of the scattered electric and magnetic dipole fields and E_x^0 and H_y^0 are the x and y components of the electric and magnetic fields incident upon the phase center of the electric-magnetic dipole scatterer. (In the far field, $\mathbf{E} = -b_e[e^{ikr}/(kr)]\sin\theta_e \hat{\boldsymbol{\theta}}_e$ and $\mathbf{H} = -b_m Y_0 [e^{ikr}/(kr)] \sin\theta_m \hat{\boldsymbol{\theta}}_m$, where θ_e and θ_m are measured from the positive electric and magnetic dipole moment axes, respectively.) The analysis is restricted to scatterers that are sufficiently small that only electric- and magnetic-dipole scattered fields are significant or to frequencies for which all the scattered multipole fields are negligible except the electric- and magnetic-dipole fields.

For an infinite 1D periodic array of these orthogonal electric and magnetic dipoles perpendicular to the array axis and separated by a distance d, the following transcendental equation for the propagation constants β of the

traveling waves supported by this array results from equating the electric and magnetic fields incident on any one electric-magnetic dipole scatterer to the sum of the electric and magnetic fields scattered by all the other electric-magnetic dipoles [7]

$$\frac{(kd)^3 - S_e\Sigma_1}{S_e\Sigma_2} = \frac{S_m\Sigma_2}{(kd)^3 - S_m\Sigma_1} \tag{9}$$

where

$$\Sigma_1 = -(kd)^2 \ln\left[2\left(\cos kd - \cos\beta d\right)\right] - (kd)\left[F(kd+\beta d) - F(\beta d - kd)\right] - \left[G(kd+\beta d) + G(\beta d - kd)\right] - i\frac{2}{3}(kd)^3 \tag{10}$$

$$\Sigma_2 = -(kd)^2\left[\ln\sin\left(\frac{kd+\beta d}{2}\right) - \ln\sin\left(\frac{\beta d - kd}{2}\right)\right] - (kd)\left[F(kd+\beta d) + F(\beta d - kd)\right] \tag{11}$$

and it can be shown that the left- and right-hand sides of (9) are real.

Figure 2 shows the lowest branch of the kd–βd diagram computed from (11) for an infinite 1D periodic array of lossless magnetodielectric spheres with radius a equal to $.45d$ and relative permittivity and permeability given by $\epsilon_r = \mu_r = 20$. For magnetodielectric spheres, $S_e = -(3/2)ib_1^{sc}$ and $S_m = -(3/2)ia_1^{sc}$, where b_1^{sc} and a_1^{sc} are the electric- and magnetic-dipole Mie scattering coefficients, respectively. The first electric and magnetic dipole resonances of the spheres occur at $ka = .214$, which corresponds to $kd = .475$ in Figure 2. The second branch (not shown) of this kd–βd diagram is centered about the second electric and magnetic dipole resonances that occur at $kd = .813$; and so on.

TWO- AND THREE-DIMENSIONAL PERIODIC ARRAYS

The kd–βd diagrams for 2D periodic arrays of acoustic monopoles can be found straightforwardly with the help of the formula [8]

$$\sum_{m=-\infty}^{\infty} \frac{e^{ik\sqrt{(md-x)^2+\rho^2}}}{\sqrt{(md-x)^2+\rho^2}} = \frac{\pi i}{d}\sum_{m=-\infty}^{\infty} H_0^{(1)}(k_m\rho)\, e^{i(2\pi/d)mx} \tag{12}$$

which expresses the fields of an infinite line of acoustic point sources in terms of an infinite sum of discrete cylindrical waves. To find the kd–βd diagrams for 3D periodic arrays of acoustic monopoles, we also need the corresponding formula [8]

$$\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{e^{ik\sqrt{(md-x)^2+(ld-y)^2+z^2}}}{\sqrt{(md-x)^2+(ld-y)^2+z^2}} = \frac{2\pi i}{d}\sum_{m=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{e^{i(2\pi/d)(mx+ly)}\, e^{ik_{ml}|z|}}{k_{ml}d} \tag{13}$$

where $k_{ml} = \sqrt{k^2 - (2\pi m/d)^2 - (2\pi l/d)^2}$ is positive real (positive imaginary) according as $(kd)^2 > (<)\ (2\pi)^2(m^2 + l^2)$. This equation (13) expresses the fields of an infinite plane of acoustic point sources in terms of an infinite sum of discrete plane waves. Taking the curl once and twice of (12) and (13) multiplied by $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, or $\hat{\mathbf{z}}$ gives similar formulas for the electric and magnetic fields of infinite lines and planes of electric and magnetic dipoles oriented is the x, y, and z directions.

In particular, applying these formulas to 3D periodic arrays of lossless coupled orthogonal electric and magnetic dipoles oriented perpendicular to a rectangular coordinate (x, y, or z) direction of propagation and separated on a cubic lattice by a distance d, we obtain a transcendental equation in the same form as (9) but now with Σ_1 and Σ_2 given by [8]

$$\Sigma_1 = -2\pi(kd)\frac{\sin kd}{\cos\beta d - \cos kd} - 4\pi\sum_{n=1}^{\infty}\cos(n\beta d)\sum_{\substack{m=-\infty\\(m,l)}}^{\infty}\ \neq\ \sum_{\substack{l=-\infty\\(0,0)}}^{\infty}\left[(2\pi m)^2 - (kd)^2\right]\frac{e^{-n\sqrt{(2\pi)^2(m^2+l^2)-(kd)^2}}}{\sqrt{(2\pi)^2(m^2+l^2)-(kd)^2}}$$

$$-2\pi(kd)^2\sum_{l=1}^{\infty}Y_0(lkd) - 8\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left[(2\pi m)^2 - (kd)^2\right]K_0\left(l\sqrt{(2\pi m)^2 - (kd)^2}\right) + 4(kd)F(kd) + 4G(kd) - i\frac{2}{3}(kd)^3 \tag{14}$$

$$\Sigma_2 = -2\pi(kd)\frac{\sin\beta d}{\cos\beta d - \cos kd} - 4\pi(kd)\sum_{n=1}^{\infty}\sin(n\beta d)\sum_{\substack{m=-\infty\\(m,l)}}^{\infty}\ \neq\ \sum_{\substack{l=-\infty\\(0,0)}}^{\infty} e^{-n\sqrt{(2\pi)^2(m^2+l^2)-(kd)^2}}\,. \tag{15}$$

(Rapidly convergent formulas [9, sec. 8.522] exist to efficiently evaluate $\sum_{l=1}^{\infty} Y_0(lkd)$.)

Figure 3 shows the lowest branch of the βd–ka diagram computed from (9) with Σ_1 and Σ_2 inserted from (14) and (15) for an infinite 3D periodic array of lossless magnetodielectric spheres with radius a equal to $.4924d$ and relative permittivity and permeability given by $\epsilon_r = \mu_r = 20$. (Unlike 1D and 2D arrays, fast waves on lossless 3D arrays are not attenuated.) Figure 4 plots the effective relative permittivity ϵ_r^{eff} and permeability μ_r^{eff} versus ka obtained from the formula $\epsilon_r^{eff} = \mu_r^{eff} = \text{sign}[dk/d\beta](a/d)\beta d/(ka)$. We choose $a/d = .4924$ and ka along the horizontal axis in Figures 3 and 4 in order to display the good agreement (except near $ka = .2$) with the corresponding approximate results obtained by Holloway *et al.* [10, fig. 7].

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