Measurement of the Strain Induced Coefficient of Permittivity of Sapphire using Whispering Gallery Modes Excited in a High-Q Acoustic Sapphire Oscillator

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ABSTRACT

A low loss monolithic sapphire has been developed for use as a novel transducer via the interaction between the electrical and mechanical resonances. Sapphire is an excellent material for use as a low noise electro-mechanical transducer due to its low mechanical and electrical losses. The sapphire sample under investigation is a single crystal of HEMEX grade 50mm in diameter and 100mm in length, with the c-axis aligned with the z-axis of the cylinder. We utilize high quality factor Whispering Gallery (WG) modes that have a high azimuthal number to improve electromagnetic confinement, which reduces the conductor loss due to the cavity.

The sapphire cylinder is suspended about its midpoint by a single loop of wire to allow it to vibrate with minimal losses in its fundamental acoustic mode. The wire is in turn attached to a second stage of vibration isolation consisting of a two stage mass-spring system. The sapphire and suspension arrangement are situated in a cryogenic environment, and excitation of the acoustic resonance is by means of a mechanical relay switch that strikes the sapphire upon an end face. Modulation of electromagnetic modes occurs when the sapphire bar oscillates in its fundamental acoustic mode of vibration, causing dimensional changes to the crystal. This leads to a twofold mechanism of modulation of the WG resonances; firstly, the dimension is altered changing the boundary conditions of the WG resonance, and secondly, the permittivity is altered due to strain induced by the dimension change. The latter is the dominant effect.

High electrical and mechanical quality factors are obtained at low temperatures (4.2K), allowing us to operate in a regime where parametric interactions dominate. In deriving the displacement sensitivity of the Monolithic Sapphire Transducer, the acoustic mode shape and WG field distribution must be taken into account, rather than the use of a simple mass-spring model. With the aid of this model we determine for the first time the strain induced coefficient of permittivity for sapphire, both perpendicular and parallel to the c-axis. By comparison with other work it has been determined that changes in the dielectric constant due to strain are approximately eight times smaller than changes caused by thermal expansion.

INTRODUCTION

Sapphire is an excellent material for use as a low noise electro-mechanical transducer due to its low mechanical and electrical losses. Sapphire is an anisotropic material, with different components of complex permittivity and coefficients of thermal expansion parallel and perpendicular to the crystal axis (c-axis.) The MST under investigation is a single crystal of HEMEX grade 50mm in diameter and 100mm in length, with the c-axis aligned with the z-axis of the cylinder [1].

Sapphire has a low dielectric constant (of the order of 10), resulting in large radiation losses for low order modes, which results in the difficulty in trapping energy within a sapphire oscillator. This is overcome through using a class of higher order modes called “Whispering Gallery” (WG) modes [2]. WG modes are created by two counter-propagating electro-magnetic waves, which exist via total internal reflection in a dielectric material, which in general is rotationally symmetric. Pure, monolithic crystal sapphire has an exceptionally high quality factor (Q-factor) when supporting these WG modes.

The MST supports two classes of WG modes; a quasi-TM mode has the dominant electric field parallel to the crystal axis, while a quasi-TE mode has a dominant axial magnetic dependence. In reality, all WG resonances are in fact hybrids of both TE and TM modes. The parallel and perpendicular filling factors

(p_{ez}, p_{er}) are a measure of the fraction of the WG mode energy which is concentrated parallel and perpendicular to the crystal axis. The nomenclature $TE_{m,n,p+\delta}$ is employed within this paper, where the integers 'm', 'n' and 'p' are the number of azimuthal, radial and axial variations of the resonance, respectively, and δ is a number slightly less than 1. Mode "families" are groups of WG modes with the same radial and axial field variations.

We utilize high quality factor (Q-factor) WG modes that have a high azimuthal number to improve electromagnetic confinement, which reduces the conductor loss due to the cavity. Coupling to the WG field is via the radiative evanescent field outside sapphire. Modulation of WG modes occurs when the sapphire bar oscillates in its fundamental acoustic mode of vibration, causing dimensional changes to the crystal. This leads to a twofold mechanism of modulation of the WG resonances; firstly, the dimension is altered changing the boundary conditions of the WG resonance, and secondly, the permittivity is altered due to strain induced by the dimension change. The latter is the dominant effect. This can be modelled as a resonant inductor/capacitor/resistor circuit coupled to a mass/spring system, in which the oscillating mass changes the capacitance of the circuit [3-5].

METHOD TO DETERMINE THE STRAIN DEPENDENCE OF PERMITTIVITY

The displacement sensitivity *df/dz* is a quantitative measure of how much the WG mode frequency (*f*) changes due to a change in length of the sapphire rod (*z*.) Previously, a simple model has been used to approximate *df/dz* [6, 7]. The more advanced and appropriate calculation presented in this paper requires knowledge of the fraction of energy in the axial and radial directions, which is defined [8] by the electric energy filling factor (p_e);

$$p_e = \frac{W_d}{W_t} = \frac{\iiint\limits_{V_s} \varepsilon_s \underline{E}.\underline{E}^* dv}{\iiint\limits_{V} \varepsilon(v)\, \underline{E}.\underline{E}^* dv} \qquad (1)$$

Here W_d is the electric energy stored in the dielectric, W_t the total electric energy stored in the whole resonant structure, ε_s is the permittivity of the dielectric, $\varepsilon(v)$ is the relative spatially dependent permittivity of the whole resonant structure, and $\underline{E}$ and $\underline{E}^*$ are the complex electric field and its conjugate, respectively.

The fraction of electric energy in the radial direction and z-direction are denoted p_{er} and p_{ez} respectively. The dimensional filling factors p_D and p_L (where D is the diameter and L the length of the MST cylinder) are the fraction of electromagnetic energy that propagates in the radial and axial direction, respectively. These parameters may be determined from the incremental frequency rule [9];

$$p_{er} = 2\left|\frac{\partial f_{res}}{\delta \varepsilon_r}\right|\frac{\varepsilon_r}{f} \qquad p_D = \left|\frac{\partial f_{res}}{\delta D}\right|\frac{D}{f}$$
$$p_{ez} = 2\left|\frac{\partial f_{res}}{\delta \varepsilon_z}\right|\frac{\varepsilon_z}{f} \qquad p_L = \left|\frac{\partial f_{res}}{\delta L}\right|\frac{L}{f} \qquad (2)$$

Following the treatment used in [8], we have used a mode matching technique to determine the filling factors for the MST and a separation of variable technique [7] used to calculate the fields.

The total displacement sensitivity *df/dz* is obtained from the sum of partial derivatives in equation 2, with the radial component (*r*) coupled to the axial (*z*) via Poisson's Ratio, which is 0.3 for sapphire.

$$\frac{z}{f}\frac{df}{dz} = \frac{0.3}{2} p_{er} K_{\varepsilon r} - \frac{1}{2} p_{ez} K_{\varepsilon z} + 0.3 p_D - p_L \qquad (3)$$

Where the unknowns to be found are

$$K_{\varepsilon r} = \frac{d\varepsilon_r}{dr}\frac{r}{\varepsilon_r} \quad \text{and} \quad K_{\varepsilon z} = \frac{d\varepsilon_z}{dz}\frac{z}{\varepsilon_z} \tag{4}$$

In order to calculate the displacement sensitivity (*df/dz*), the parameters $K_{\varepsilon r}$ and $K_{\varepsilon z}$ must be determined [10]. Thus, if one measures the displacement sensitivity of a known TE and TM mode nearby in frequency, two equations relating to equation 3 may be solved simultaneously to uniquely determine the strain parallel and perpendicular to the crystal axis given by equation 4, given the fact that p_D, p_L, p_{er} and p_{ez} can be calculated numerically.

The above analysis to determine the strain dependence of permittivity is still inadequate, as it assumes the acoustic mode resonance produces a linear strain along the axis of the rod and that the electric field is constant throughout the crystal. In actuality the acoustic mode shape is non-linear (it is a cylinder that stretches more at the midpoint and less towards the ends) and the electric field is not uniform throughout. This will lead to an altered displacement sensitivity, which has been calculated using Finite Element Modelling, the results of which are omitted for brevity.

EXPERIMENTAL MEASUREMENTS TO VALIDATE CALCULATIONS

The experimental set-up is described in detail in [6, 11]. To briefly summarise, the MST is suspended about its midpoint by a single loop of wire to allow it to vibrate with minimal losses in its fundamental acoustic mode. The wire is in turn attached to a second stage of vibration isolation consisting of a two stage mass-spring system of copper masses (mass 2kg each) and low frequency springs. The MST and suspension arrangement are situated in a cryogenic environment, and excitation of the acoustic resonance is by means of a mechanical relay switch that strikes the sapphire upon an end face.

A low phase noise pump oscillator signal is coupled to the WG mode via a microwave probe, and the mechanical resonance added sidebands to the reflected signal at $\pm\omega$ offset from the microwave carrier. The signal was then mixed with a phase shifted portion of carrier signal which was adjusted such that the system was phase sensitive.

The absolute value of *df/dz* cannot be measured directly without using a detailed calibration method for a parametric transducer [12, 13]. However, we have verified the mode shape calculations by taking relative measurements of the magnitudes of *df/dz*'s between modes with respect to the $TM_{14,1,\delta}$ mode. This is possible because the mechanical Q-factor is high and allows us the possibility of switching the readout system between WG modes and comparing the signal output without significant changes occurring in the vibrational amplitude of the oscillator.

The ratios of measured *df/dz*'s between WG modes are in agreement with those calculated. From knowledge of the displacement sensitivities of both TM and TE modes and calculations of the filling factors, equation 3 can be used to determine the strain dependence of permittivity. A solution of simultaneous equations will yield $K_{\varepsilon r}$ and $K_{\varepsilon z}$, and gives us;

$$\frac{1}{\varepsilon_z} d\varepsilon_z \approx 4.2 \pm 0.3 \frac{1}{L} dL$$

and

$$\frac{1}{\varepsilon_r} d\varepsilon_r \approx 3.1 \pm 0.3 \frac{1}{r} dr$$

This result is the strain induced in a sapphire oscillator causing a fractional change in the dielectric constant.

Work carried out at UWA by Hartnett et al. [14] that measured the change dielectric constant due to temperature (as opposed to strain) reports;

$$\frac{1}{\varepsilon}\frac{d\varepsilon}{dT} \approx 32 \frac{1}{D}\frac{dD}{dT}$$

It would suggest that changes in resonator dimensions due to strain mediates changes in the dielectric constant with a factor that is eight to ten times smaller than those changes caused by thermal expansion.

CONCLUSION

Parametric interaction has been observed in a sapphire monocrystal with high electric and acoustic quality factors in a sapphire monocrystal. The acoustic mode shape and electromagnetic field distribution have been taken into account in the modelling of the interaction. Accurate measurements of the dynamics of the monocrystal has yielded information on the strain induced coefficient of permittivity for sapphire, and with selection of either a TM or TE mode, the anisotropy of this parameter has been measured.

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