SOLUTION AND STABILITY ANALYSIS OF A NEW INTEGRAL EQUATION FOR THE TRANSIENT SCATTERING BY A FLAT, RECTANGULAR CONDUCTING PLATE

Jos G.A. van Riswick[1] **and Anton G. Tijhuis**[2]

[1] *Eindhoven University of Technology, Faculty of Electrical Engineering,*
P.O. Box 513, 5600 MB Eindhoven, the Netherlands, e-mail: J.G.A.v.Riswick@tue.nl

[2] *As (1) above, but e-mail: A.G.Tijhuis@tue.nl*

ABSTRACT

In this contribution, we present a new Hallén-type formulation for scattering by a two-dimensional, perfectly conducting rectangular plate. We derive two coupled integral equations for the components of the induced surface current, which are solved by marching on in time. The numerical results may exhibit instabilities, but the solution can be stabilized in various ways. To arrive at a more detailed understanding of the stability problem, we reformulate the computation in a combination of a finite-difference time-domain computation and the solution of two scalar integral relations which no longer contain space and time differentiations. Both procedures are coupled via the boundary conditions at the edges of the plate. As a reference result, we consider the transient excitation of an infinite flat plate by a pulsed, vertical electric dipole, for which the unknown vector potential and the induced surface currents are available in closed form.

INTRODUCTION

The electric-field integral equation (EFIE) is a common tool for determining scattered or radiating fields emanating from a perfectly conducting, open surface. In this type of approach the induced surface current and/or charge density are solved from the integral equations, and, subsequently, the scattered electromagnetic field is determined at any given point by evaluating the integrals in a conventional integral representation. Advantages of this approach are that the radiation condition is accounted for inherently, and that only two-dimensional surface currents and/or charges need to be computed over the area of the scattering surface, instead of three- dimensional electromagnetic fields over a volume surrounding that surface.

When the EFIE is solved directly in the time domain, instabilities are often observed in the computed results. Numerical experiments for a number of two-dimensional transient-scattering problems [1] as well as for perfectly conducting surfaces indicate that the occurrence of such unstable solutions is enhanced by the presence of space derivatives in these equations. In particular, the manner in which these derivatives are approximated in the space discretization has a considerable influence on the stability of the computational solutions. To investigate this observation in more detail, we consider the "canonical" problem of a rectangular, flat plate excited by a pulsed incident field. The role of the space derivatives is isolated from the complete computation by reformulating the procedure in three steps. First, we reduce the original EFIE to a Hallén-type integral equation, which consists of two coupled, scalar integral equations for the transverse currents $J_x(x,y,t)$ and $J_y(x,y,t)$. This step was inspired by the success of using Hallén's integral equation for the total current $I(z,t)$ along a straight conducting thin wire in time-domain computations [2]. Marching-on-in-time schemes based on our new integral equation are still unstable, but can be stabilized with the aid of time averaging.

Therefore, the second modification in our approach is to reformulate the integral equations in terms of two coupled differential equations that can be solved by a finite-difference time-domain scheme, and two uncoupled scalar integral equations, where the current densities $J_x(x,y,t)$ and $J_y(x,y,t)$ are resolved from the corresponding components of a vector potential $\boldsymbol{A}(x,y,t)$. Here, we found our inspiration in a similar treatment of the EFIE in [3]. A straightforward discretization of the differential equations results in a scheme that satisfies the well-known Courant stability criterion for two-dimensional wave propagation. The scalar integral equation is then solved by marching on in time, as described in [3]. For the case where the flat plate is infinite, both parts of the computation can be validated independently, by comparing numerical results with closed-form expressions for the field excited by a horizontally or vertically polarized, pulsed electric point dipole located above the plate. When the plate has finite dimensions, coupling occurs via the boundary conditions at its edges. This means that the two parts of the computation must be combined, and that the stability of the entire scheme depends on the implementation of these boundary conditions.

FORMULATION OF THE PROBLEM

A pulsed plane wave is incident on a two-dimensional, perfectly conducting rectangular plate as shown in Fig. 1. The Cartesian coordinate system is chosen such that the plate is located at $0 < x < a$, $0 < y < b$, and $z = 0$. The surrounding homogeneous, linearly and instantaneously reacting, isotropic medium has permittivity ε and permeability μ. We start from the standard form of the electric-field integral equation

$$\left(\nabla_T \nabla_T \cdot - \frac{1}{c^2}\partial_t^2\right) \boldsymbol{A}(\boldsymbol{r}_T, t) = -\varepsilon \partial_t \boldsymbol{E}_T^i(\boldsymbol{r}_T, t), \qquad (1)$$

which holds for $\boldsymbol{r}_T$ on the plate. In (1), the subscript $_T$ stands for transverse, and the vector potential $\boldsymbol{A}(\boldsymbol{r}_T, t)$ is related to the unknown surface current density $\boldsymbol{J}_S(\boldsymbol{r}_T, t)$ in the plane $z = 0$ according to

$$\boldsymbol{A}(\boldsymbol{r}_T, t) = \int_0^a dx' \int_0^b dy' \, \frac{\boldsymbol{J}_S(\boldsymbol{r}_T', t - R/c)}{4\pi R}, \qquad (2)$$

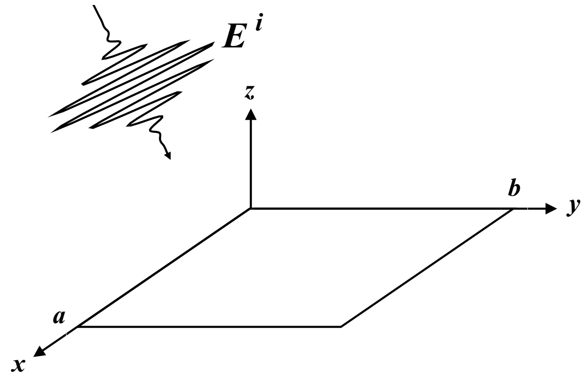

Fig. 1. Pulsed plane wave incident on a flat rectangular plate.

with $R = |\boldsymbol{r}_T - \boldsymbol{r}_T'|$ and $c = 1/\sqrt{\varepsilon\mu}$. The aim of the computation is to determine $\boldsymbol{J}(\boldsymbol{r}_T, t)$ for $0 < x < a$ and $0 < y < b$ for a given behavior of the transverse incident field $\boldsymbol{E}_T^i(\boldsymbol{r}_T, t)$. Once this surface current is known, it can be treated as a "secondary" source that generates the electromagnetic field scattered by the plate.

HALLÉN-TYPE INTEGRAL EQUATION

As mentioned in the introduction, numerical differentations with respect to space and time are often the source of instabilities in time-domain computations. Therefore, they should be avoided as much as possible. To this end, we break up (1) into its x- and y-components. Since we restrict ourselves to the plane $z = 0$, we may replace $\boldsymbol{r}_T$ by the corresponding transverse coordinates x and y. This reduces (1) to a pair of coupled scalar electric-field integral equations:

$$\left(\partial_x^2 - \frac{1}{c^2}\partial_t^2\right) A_x(x,y,t) + \partial_x \partial_y A_y(x,y,t) \quad = \quad -\varepsilon \partial_t E_x^{\mathrm{inc}}(x,y,t), \qquad (3)$$

$$\left(\partial_y^2 - \frac{1}{c^2}\partial_t^2\right) A_y(x,y,t) + \partial_y \partial_x A_x(x,y,t) \quad = \quad -\varepsilon \partial_t E_y^{\mathrm{inc}}(x,y,t). \qquad (4)$$

In analogy with the case of a straight thin-wire segment, we identify the differential operators in the first terms on the left-hand sides as one-dimensional wave operators, that can be evaluated with the aid of a one-dimensional Green's function technique. This results in

$$\begin{aligned} A_x(x,y,t) \quad &= \quad -\frac{1}{2}\int_0^a dx' \mathrm{sgn}(x-x')\, \partial_y \, A_y(x', y, t - |x-x'|/c) \\ &\quad + \frac{Y}{2}\int_0^a dx' E_x^{\mathrm{inc}}(x', y, t - |x-x'|/c) + F_0(t - x/c, y) + F_a(t - (a-x)/c, y), \\ A_y(x,y,t) \quad &= \quad -\frac{1}{2}\int_0^b dy' \mathrm{sgn}(y-y')\, \partial_x \, A_x(x, y', t - |y-y'|/c) \\ &\quad + \frac{Y}{2}\int_0^b dy' E_y^{\mathrm{inc}}(x, y', t - |y-y'|/c) + G_0(t - y/c, x) + G_b(t - (b-y)/c, x). \end{aligned} \qquad (5)$$

In (5), the time signals $F_0(t, y)$, $F_a(t, y)$, $G_0(t, x)$ and $G_b(t, x)$ correspond to homogeneous solutions of the relevant one-dimensional wave equations. To fix these signals, we need to augment the pair of integral equations in (5) with the boundary conditions

$$J_x(0,y,t) = 0, \quad J_x(a,y,t) = 0, \quad J_y(x,0,t) = 0, \quad J_y(x,b,t) = 0, \qquad (6)$$

which follow from the observation that $\boldsymbol{J}_S(x,y,t)$ is tangential at the edges of the plate.

Our first approach was to solve the system of equations formed by (5) and (6) by *marching on in time*. In the space discretization, the plate is subdivided into $L \times M$ cells with dimensions $\Delta x = a/L$ and $\Delta y = b/L$. The unknown currents are approximated by rooftop functions, and the integral equation is collocated on a staggered grid, at $x = \ell\Delta x$ and $y = (m + \frac{1}{2})\Delta y$, and $x = (\ell + \frac{1}{2})\Delta x$ and $y = m\Delta y$, respectively. This involves approximations of the form

$$J_x(x,y,t) = \sum_{\ell=1}^{L-1} \sum_{m=0}^{M-1} J_x[\ell, m; t]\, \Lambda(x - \ell\Delta x; \Delta x)\, \Pi(y - [m + \tfrac{1}{2}]\Delta y; \Delta y), \qquad (7)$$

where $\Delta x = a/L$, $\Delta y = b/M$, $\Lambda(\xi; h)$ is a triangle function of width $2h$ and $\Pi(\xi; h)$ is a rectangle function of width h. In (7), the first two boundary conditions of (6) are accounted for by leaving out the terms with $\ell = 0$ and $\ell = L$. Similarly, its counterpart for $J_y(x, y, t)$ satisfies the last two conditions of (6). In both cases, the relevant integral equation from (5) is collocated as well at these points to allow the determination of the unknown time signals. Finally, the time coordinate is discretized with time step Δt. The time retardation in the the delayed current $J_x(x, y, t - R/c)$ is handled by linear interpolation in time, and the differentiations with respect to y and x in (5) are evaluated via integration by parts. The resulting integrals over x' and y' in each cell can now be evaluated in closed form. The first integrals over x' and y' in the right-hand sides of (5) are approximated by a repeated midpoint rule, while the second ones are evaluated in closed form. Depending on the choice of the time step, the space-time discretization described above results in an explicit or an implicit scheme. In the latter case, a sparse matrix equation must be solved at each discrete time step.

As in the case of the original time-domain equation (1), the solution may exhibit instabilities. In our opinion, these instabilities can be explained from the fact that, by choosing a fixed space discretization, one inherently introduces an increasing error for increasing frequencies. Especially near the Nyquist frequency, this may lead to the introduction of spurious poles outside the unit circle in the discretized-frequency plane. The corresponding residual contributions are alternating, exponentially increasing time sequences. Based on this interpretation, several procedures seem available for stabilizing the solution. Until now, we have had partial success with smoothing the result after each time step, and with choosing a larger time step and using an implicit scheme. Smoothing does have a stabilizing effect, but generally affects the accuracy of the solution. Moreover, refining the discretization may bring back the instabilities. Using an implicit scheme is not as effective as we expected. The occurrence of instabilities depends critically on the condition of the system matrix of the linear system of equations that must be solved for each time step to obtain the field at that instant. This system matrix, in turn, depends directly on the manner in which the space-time integral equation is discretized.

FINITE-DIFFERENCE FORMULATION

A closed-form stability analysis of the procedure given in the previous chapter is, to our knowledge, not available. Even for the case of an infinite plate, generalizing the Von Neumann procedure that is customary for finite-difference calculations is too complicated. Therefore, we reformulate the integral equations in (5) in such a way that the vector potential can be solved from a system of differential equations with the aid of finite differences. This part of the time-marching scheme can then be subjected to a Von Neumann stability analysis. This leaves the calculation of the the induced surface currents from the calculated potential. However, at least the consequences of the numerical differentiations in the scheme can be understood in this manner. For the infinite-plate problem, the analysis provides exact answers since we do not have to deal with boundary conditions. For a finite plate, the results are identical to those for the infinite plate until the induced currents reach the edge of the spatial grid. From that instant on, boundary conditions must be imposed and the two parts of the computation can no longer be carried out independently.

To arrive at the desired scheme, we introduce the auxiliary quantities

$$\begin{aligned}
R_x(x, y, t) &= \frac{1}{2}\int_0^b dy' \,\mathrm{sgn}(y - y')\, A_x(x, y', t - |y - y'|/c), \\
R_y(x, y, t) &= \frac{1}{2}\int_0^a dx' \,\mathrm{sgn}(x - x')\, A_y(x', y, t - |x - x'|/c), \\
Q_x(x, y, t) &= \frac{1}{2}\int_0^b dy' \, A_x(x, y', t - |y - y'|/c), \\
Q_y(x, y, t) &= \frac{1}{2}\int_0^a dx' \, A_y(x', y, t - |x - x'|/c).
\end{aligned} \tag{8}$$

Partial differentiation of the first and third lines of (8) with respect to y, and of the second and fourth lines of (8) with respect to x directly results in the first-order partial differential equations

$$\begin{aligned}
\partial_y R_x(x, y, t) + \frac{1}{c}\,\partial_t Q_x(x, y, t) &= A_x(x, y, t), \\
\partial_x R_y(x, y, t) + \frac{1}{c}\,\partial_t Q_y(x, y, t) &= A_y(x, y, t), \\
\partial_y Q_x(x, y, t) + \frac{1}{c}\,\partial_t R_x(x, y, t) &= 0, \\
\partial_x Q_y(x, y, t) + \frac{1}{c}\,\partial_t R_y(x, y, t) &= 0.
\end{aligned} \tag{9}$$

The two coupled Hallén-type integral equations given in (5) now reduce to

$$\begin{aligned} A_x(x,y,t) &= -\partial_y R_y(x,y,t) + P_x(x,y,t) + F_0(t-x/c,y) + F_a(t-(a-x)/c,y), \\ A_y(x,y,t) &= -\partial_x R_x(x,y,t) + P_y(x,y,t) + G_0(t-y/c,x) + G_b(t-(b-y)/c,x), \end{aligned} \tag{10}$$

where $P_x(x,y,t)$ and $P_y(x,y,t)$ are known functions that depend on the incident field:

$$\begin{aligned} P_x(x,y,t) &= \frac{Y}{2}\int_0^a dx' E_x^{\mathrm{inc}}(x',y,t-|x-x'|/c), \\ P_y(x,y,t) &= \frac{Y}{2}\int_0^b dy' E_y^{\mathrm{inc}}(x,y',t-|y-y'|/c). \end{aligned} \tag{11}$$

The system of partial differential equations can be consistently translated into a finite difference scheme by approximating the derivatives using the central difference rule, and using a grid staggered in space and time.

In formulating the difference equations, we use a short-hand notation to indicate the approximated version of the unknown quantities, e.g.,

$$A_x[\ell,m,n] = A_x(\ell\Delta x, m\Delta y, n\Delta t), \tag{12}$$

where ℓ, m and n denote integer numbers, and where $\Delta x, \Delta y$ and Δt are the spatial cell sizes in the x- and y-directions, and the time step. To arrive at a consistent scheme, we evaluate the approximated quantities $A_x[\ell, m+\frac{1}{2}, n]$, $A_y[\ell+\frac{1}{2}, m, n]$, $R_x[\ell,m,n]$, $R_y[\ell,m,n]$, $Q_x[\ell, m+\frac{1}{2}, n+\frac{1}{2}]$, and $Q_y[\ell+\frac{1}{2}, m, n+\frac{1}{2}]$. By sampling in this manner, we arrive at an FDTD scheme on a Yee-type grid. The spatial part of the sampling is illustrated in Figure 2.

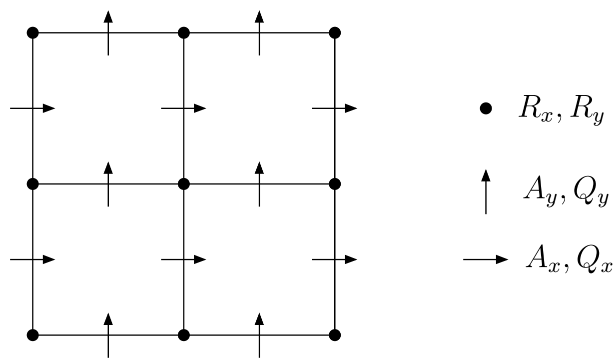

Fig.2. Four cells of the finite difference grid.

The resulting finite difference scheme can be subjected to a Von Neumann stability analysis to derive a stability condition for the scheme. We arrive at the usual Courant condition for two-dimensional propagation:

$$c\Delta t \le 1/\sqrt{1/\Delta x^2 + 1/\Delta y^2}. \tag{13}$$

As can be observed from Figure 2, the sampling of A_x and A_y is consistent with the space discretization for the original Hallén-type equations (5). This means that we can use the same procedure to discretize the equations for obtaining J_x and J_y from the x- and y-components of (2). In the space discretization, we have introduced one refinement. To ensure that the retarded time $t' = t - R/c$ is as much as possible in the correct "time zone" $(n-k-1)\Delta t < t' < (n-k)\Delta t$, we carry out the time interpolation and the integration over x' and y' in (2) over a subgrid in each cell prior to the marching-on-in-time computation. In this manner, we allow each spatial cell to cover more than one "time zone", which has been demonstrated in the acoustic case to contribute to the stability of the solution [4].

Finally, it should be mentioned that each step of the procedure described above can be verified independently for the special case of excitation by a vertical electric dipole located above the center of the plate. Until the currents reach the end of the plate, the solution is identical to that for an infinite plate, for which $A_{x,y}$ and $J_{x,y}$ are available in closed form. This means that each step of the procedure can be validated independently. Results cannot be included here because of space limitations, but will be presented and discussed at the conference.

REFERENCES

[1] A.G. Tijhuis, *Electromagnetic Inverse Profiling; Theory and Numerical Implementation*, VNU Science Press, Utrecht, the Netherlands, 1987, Sections 3.2 and 3.5.

[2] A.G. Tijhuis, Z.Q. Peng and A. Rubio Bretones,"Transient excitation of a straight thin-wire segment: a new look at an old problem", *IEEE Trans. Antennas Propagat.*, vol. 40, 1132-1146, 1992.

[3] B.P. Rynne and P.D. Smith, "Stability of time marching algorithms for the electric field integral equation", *J. of Electromagnetic Waves and Applications*, vol. 4, 1181-1205, 1990.

[4] G.C. Herman, "Scattering of transient acoustic waves by an inhomogeneous obstacle", *J. Acoust. Soc. Am.*, vol. 69, 909–915, 1981.