A METHOD TO OBTAIN KOCH FAMILY GRATINGS AND STUDY OF THE ELECTROMAGNETIC SCATTERING

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ABSTRACT

The mathematical method for representing complex objects is important for the study of the fine structure in the Fraunhofer region. The mathematical foundation for these cases is related with the intersection between sets. The mean consideration in this paper is the construction of fractal sets, such as the Koch snowflake, through the union operation. In each case different aspects of the obtained diffraction pattern are shown. Also, the contribution from the simple components of the structures are taken into account for future applications in dynamic optical processing.

INTRODUCTION

The geometric properties of fractal objects should be reflected, in any way, in the result of the interaction of this object with the an electromagnetic beam coming from a source, for example a laser. Circular and rectangular apertures can be studied using simple methods, and many authors considered different aspects of such problem, where the diffraction along the optical axis and the non-paraxial region were analyzed. But, apertures with a complex structure have not been mostly used [1,2].

Some examples, in which we are particularly interested, are: 1) diffraction by periodic and quasi-periodic gratings, 2) scattering from rough surfaces and 3) propagation of waves in inhomogeneous and non-linear media. Also, of great importance is the extension of these results to structures with a more complex geometry, such as fractal objects , whose properties in the scattering of electromagnetic waves have been studied by several authors. In the three cases mentioned before the characterization has been developed using different numerical techniques and experimental arrays, through the intensity distribution which is obtained from the interaction of waves with the structure under consideration.

There are several studies which report interesting properties of the diffracted fields generated by fractal objects. Allain and Cloitre [3] are among the first to study some properties of the Fraunhofer diffraction with an experimental setup, for the case of Cantor bars as well as for Vicsek and Sierpinski fractals. Asakura et al. [4] used a correlation function in order to quantify diffraction properties from Cantor fractal gratings, in the Fresnel and Fraunhofer region. Also, Jaggard et al. [5] have developed some interesting works about serrated apertures.

To obtain a fractal curve, we begin with the initiator and we apply the generator successively. For the Koch snowflake, the initiator can be a triangle or a hexagon, for the two-dimensional case, and the generator can be applied on each side.

Our interest is to treat the relationship between the geometry of the object and the corresponding intensity distribution in the scattered fields. The mathematical method for representing complex objects is important for the study of the fine structure in the Fraunhofer region. At the present work we make the necessary considerations about the construction method that facilitates us the study of the scattered fields based on more simple structures, such as periodic distributions which were widely used in classical electromagnetic theory. The mathematical foundation for these cases is related with the intersection and union between sets.

The intensity distributions are characterized with a cardinal sine function as an envelope, according with the method for building the objects.

The study of diffraction by Koch apertures can easily be extended to random and quasi-random gratings. This could bring, as a consequence, the definition of new similarity functions to estimate the relationship between statistical parameters. These properties and the possibility of manipulating the periodical components to be filtered, represent useful results for applications in optical metrology, computer-generated holograms and image processing. The possibility of being able to obtain directly the Fourier transform of a complex object represents by itself an important result for manipulation of these results for optical processing.

The intensity diffracted for these cases are included in the results obtained, where we can also appreciate the directions of light distribution, which are directly related with the structure of the initiator and with the rectangular components. We have three directions for the case of triangular initiator and four directions when there are two squares with different

angular positions [6]. This way, the distribution of intensity in the diffraction, at the proximity of the Fraunhofer region can be modified through the manipulation of periodic components of the Koch snowflake, in the same way that for the case of the Sierpinski carpet. This can be made, for example, through the filtering or quasi-randomization of such periodic components, similarly to the cases studied for Cantor diffraction gratings.
A simple mathematical expression is used to represent the transmittance through the superposition of two-dimensional periodic functions. This work shows the possibility of using fractal gratings with different fine structure, or variable fine structure in each step of the iteration, which can be used for optical information processing. The form of such fine structure can be controlled through the parameters of single function and we can applied the results obtained from this paper for dynamic processing, when we have signals which are variable with the time. These properties and the possibility of manipulating the periodical components to be filtered, represent useful results for applications in optical metrology and image processing.
All the characteristic properties studied here can be applied to the determination of geometric structures in the cases of particles [7] and image processing, which are of great interest for us due their relationship with the study of biological systems.
In the future, a more detailed study of variable geometry and the contribution of the fine structure is expected, as well as the deeper studies in the Fresnel and Fraunhofer regions since, interesting applications of the self-imaging phenomena are arising due to the superposition of periodical functions and the possibility of measurement of other parameters in the structure. Also, the contribution of simple components from the structures is taken into account for future applications in dynamic optical processing.

MATHEMATICAL BASIS AND RESULTS OBTAINED

Diffraction at the Fraunhofer region

The field in the Fraunhofer region, for a two-dimensional transmittance in a general form, is given by:

$$U(x', y') \propto F\{U(x, y)\} \tag{1}$$

being *U(x,y)* the electromagnetic field on the object and *F{}* is the Fourier transform.

Construction of Koch snowflakes

We define a transformation A on the set E belonging to the class of compact non-empty sets *D*. If the union set:

$$\mathsf{F} = \bigcup_{i=0}^{\infty} \mathsf{A}^i \tag{2}$$

have the property that, for a given order *i*:

$$X \cap C\left[A^i(E)\right] = A^i(E) / A^{i-1}(E) \neq \phi \tag{3}$$

then, the F set is invariant. Where $C[A^i(E)]$ denote the complementary set $A^i(E)$.
Then, the characteristic function for an invariant set, with a cutting in the iterative generation, can be expressed as:

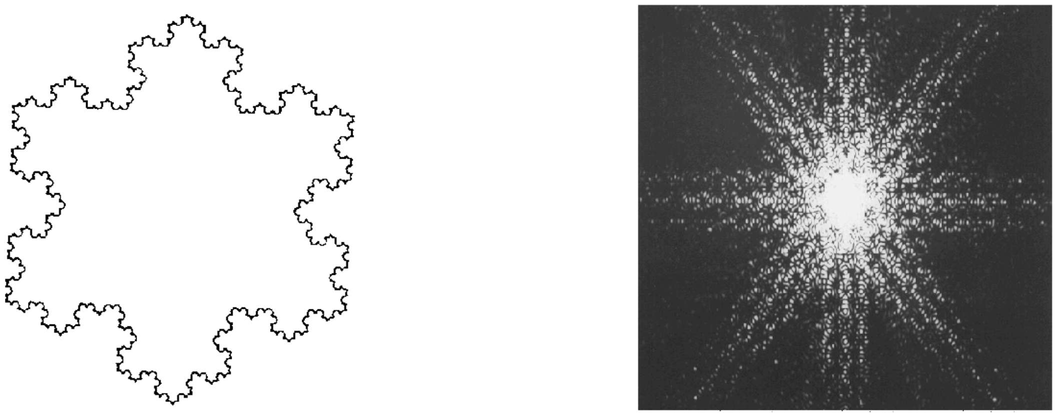
$$\chi\left[\bigcup_{i=0}^{N} A^i(E)\right] = \sum_{i=0}^{N} \chi\left[A^i(E)\right] + \sum_{r=2}^{N} (-1)^{r-1} \sum_{i_1 < \ldots < i_r}^{N} \left\{\prod_{s=1}^{r} \left\{\chi\left[A^{i_s}(E)\right]\right\}\right\} \tag{4}$$

where the number of terms in the last sum can be calculated through the permutations of N elements taken of r. Definition in Eq. (4) synthesizes mappings with magnifications and translations, but also, we must to include the rotation, for periodic distribution, as basic operation to generate fractal sets. Different structures can be obtained, in a simple way, using rectangular functions. Other operations as filtering and randomization can be applied for these cases. Fig. 1 shows an example about triangular initiator with the corresponding distribution of intensities in the diffraction from such structures. Is clear the influence of the randomization in the distribution of elements on the diffracted field, just as we prove for another cases.

The Koch fractal with triangular and square initiators can be built adding the rotation operation to the initial sets obtained with Eqs. (1) and (2), using the union operation for two fractal sets.

$$\chi[\;]=\begin{cases}\left\{\sum_{l=1}^{n} S_j\, rect\left[\dfrac{y-y_l}{\Delta_y(l)}\right]\right\}^{h} \quad , \quad h=\begin{cases}1 & \text{for triangular initiator}\\ 0 & \text{for square initiator}\end{cases}\\[2em] \left\{\sum_{\substack{i=1\\ j=1}}^{n} rect\left[\dfrac{x+\frac{\sqrt{3}}{3}y+x_i}{\Delta_x}\right] rect\left[\dfrac{x-\frac{\sqrt{3}}{3}y+x_j}{\Delta_x}\right]\right\}^{g_l}\end{cases} \tag{5}$$

being l=1,...,N, and the characteristic widths Δ_1 and Δ_2 . In the intensity diffracted we can also appreciate the directions of light distribution, which are directly related with the structure of the initiator and with the rectangular components. We have three directions for the case of triangular initiator and four directions when there are two squares with different angular positions. This way, the distribution of intensity in the diffraction, at the proximities of the Fraunhofer region can be modified through the manipulation of periodic components of the Koch snowflake, in the same way that for the case of the Sierpinski carpet [8]. This can be made, for example, through the filtering or quasi-randomization of such periodic components, similarly to the cases studied for Cantor diffraction gratings.



(a) (b)

Fig. 1. (a) Koch snowflake with triangular initiator, (b) the corresponding diffraction pattern.

CONCLUSIONS

We are shown the construction method for fractal structures using more simple structures, such as periodic distributions which are very important in optical processing. Also, an example for the diffraction at the Fraunhofer region is shown for Koch snowflake. Furthermore, the relation between the fractal grating with variable contour and the intensity diffracted from it is dependent not only of the geometric distribution of the elements into the all structure.

A simple mathematical expression is used to represent the transmittance through the superposition of two-dimensional periodic functions. This work shows the possibility of using fractal gratings with different fine structure, or variable fine structure in each step of the iteration, which can be used for optic processing of information purposes.

Starting from the developments that were obtained for different fractal structures, by using classic functions of the optical theory, interesting results have been obtained from the basic and applied point of view. Also, the study of the effect of each periodic component can be extended for diverse fractal forms. Is our interest to extend such construction methods with periodic distributions to other systems which include concepts as randomness and multifractality, that means more complex systems from the physical point of view. This is very important, since the properties of the objects will have consequences on the properties of the scattered fields.

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