Fractal Superlattices: Scattering and Inverse Scattering

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ABSTRACT

Many fractal structures, when interrogated by an electromagnetic wave, imprint a characteristic signature on the scattered field. Here we use generalized Cantor fractal superlattices as an illustration and discover that frequency-domain and time-domain analyses of the scattering data yield fractal descriptors such as the similarity fractal dimension, lacunarity, stage of growth and number of gaps for these structures. In particular, we use twist plots (frequency-domain data) and wavelet skeletons (time-domain data) to examine the "footprints" these fractals leave in the reflection data. Self-similar methods of calculation can be used to generate the scattering data by iteration for the direct scattering problem.

INTRODUCTION

We review the work in *fractal electrodynamics* [1] that involves the interaction of waves with generalized Cantor fractal superlattices. Although the initial work on scattering from a fractal distribution of layers, or superlattice, started circa 1990 [2-6], we provide an overview of our recent work [7-13] on the reflection and transmission properties of fractal superlattices. In the frequency domain, the solution to the scattering problem is formulated, solved analytically, and characterized for variations in similarity fractal dimension D (i.e., the amount of space the underlying fractal takes up), the lacunarity parameter ε (i.e., a measure of the texture or "gappiness" of the fractal), number of gaps n_{gaps} (number of gaps in first stage Cantor set), and stage of growth S (i.e., the iteration number associated with the object's fractal geometry). In the time domain, a wavelet analysis of the scattering data due to an incident impulse provides the framework for the results presented here by examining the characteristic "skeletons" in time-scale space. It has been found that the similarity fractal dimension, lacunarity and stage of growth can be retrieved from the scattered field data using this analysis.

FREQUENCY-DOMAIN RESULTS (TWIST PLOTS)

The research reported here is concerned with the reflection and transmission of electromagnetic waves from polyadic, or multi-gap, Cantor fractal superlattices. We characterize wave interactions with these structures and relate reflection data to geometrical fractal descriptors such as similarity fractal dimension D, lacunarity parameter ε, number of gaps n_{gaps}, and stage of growth S. The reflection data is generated for arbitrary angles of incidence.

The analytical solution displays the functional form of the solution of a family of fractal structures that are formed through iteration. In this way, the iterative nature of the geometry is directly related to the iterative nature of the solution and the physics of the problem is linked to its mathematics using a doubly recursive method. This computational technique efficiently provides the reflection and transmission coefficients for a large number of layers.

By stacking the scattering data as function of frequency with variations in lacunarity parameter ε, one can form twist plots as demonstrated in Fig. 1. This method of displaying reflection data allows one to observe the easily identifiable characteristic nulls which in turn can be explained from simple physical principles of destructive interference. These characteristic nulls also provide information on number of gaps n_{gaps} and stage of growth S. The similarity fractal dimension D information is embedded in both the structure of the twist plots and also in the low-frequency data. It too can be retrieved from reflection data.

For example, in Fig. 2 we display two twist plots characteristic for even- an odd-gap Cantor superlattices. The banding structure of these plots changes with number of gaps n_{gaps} and is associated with even- and odd-gap superlattices.

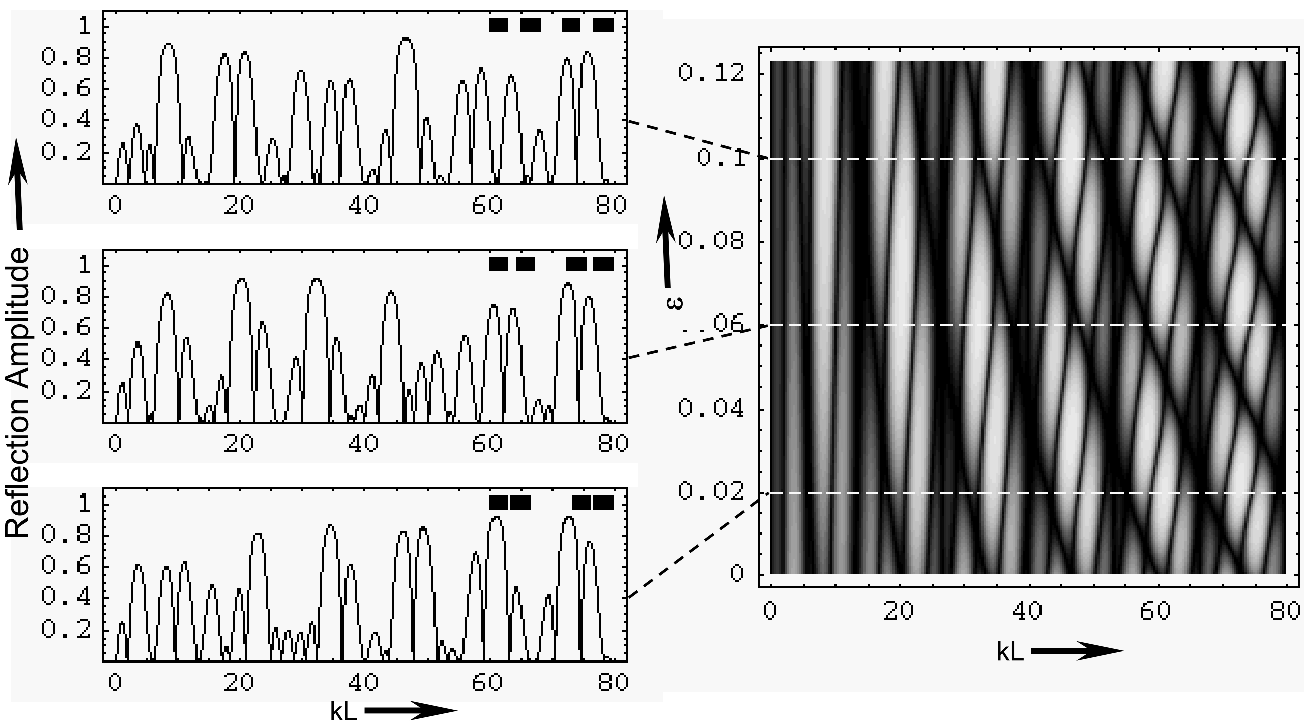

Fig. 1. Reflection amplitude for Cantor fractal superlattice (left) as a function of normalized frequency kL for increasing lacunarity parameter ε (bottom to top) and a gray-scale twist plot (right) as a continuous function of ε (black is zero). Note the characteristic set of nulls that form the structure in the twist plot. Here $D = 3/4$, $n_{gaps} = 3$ and $S = 1$.

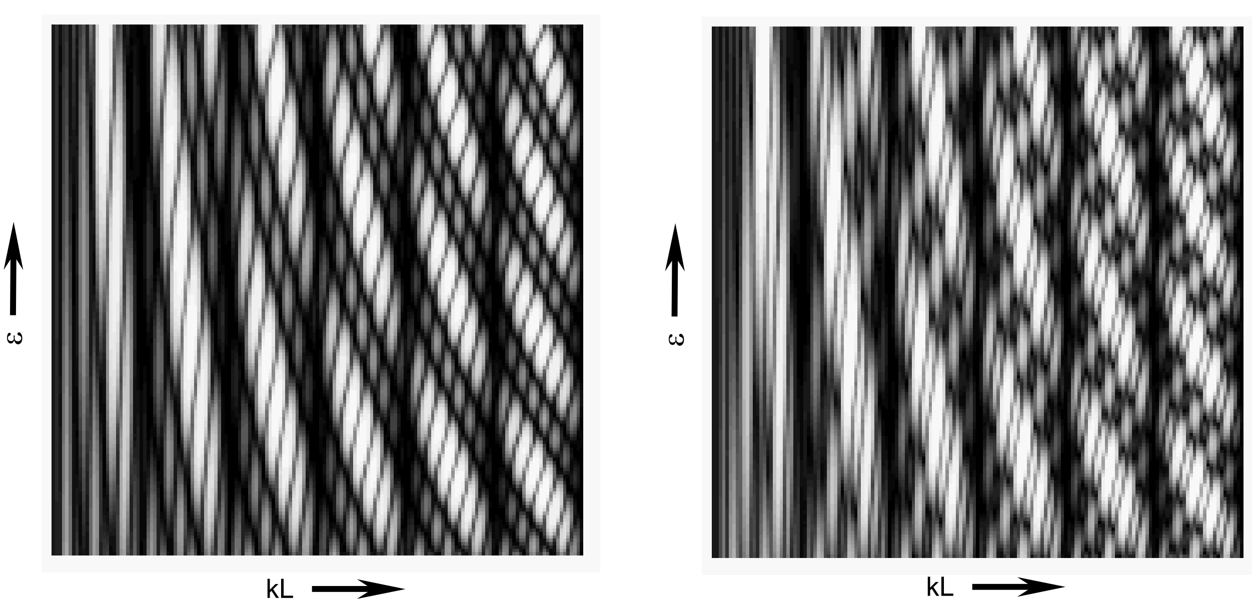

Fig 2. Example of two gray-scale twist plots displaying reflection amplitudes of generalized Cantor superlattices as a function of normalized frequency kL and lacunarity ε for an odd number of gaps ($n_{gaps} = 5$, left) and an even number of gaps ($n_{gaps} = 6$, right). There is a characteristically different banding structure for superlattices with even and odd number of gaps. Here $D = 3/4$ and $S = 1$.

TIME-DOMAIN RESULTS (WAVELET SKELETONS)

The effects of the self-similarity of Cantor superlattices may also be explored from the analysis of their impulse response. The problem consists in determining a direct relation between the primary descriptors of such objects, dimension *D*, lacunarity parameter ε, number of gaps n_{gaps}, and stage of growth *S* and the reflected time-domain signal. As illustrated in Fig. 3, due to multiple reflections, the impulse response of fractal objects exhibits a succession of abrupt changes, or singularities, in the time domain. Here we use a wavelet analysis to explore the temporal distribution of singularities in this erratic reflected signal and to relate it to the descriptors of the interrogated Cantor superlattices.

Figure 4 shows the continuous wavelet transform modulus of the impulse response shown in the insert of Fig. 3 (bottom middle). Next, Fig. 5 represents the skeleton of this transform showing the location in the time-scale domain of the modulus maxima. We observe that some maxima are distributed on a hierarchical structure (see the arch-like structure indicated by arrows in Fig. 5). As discussed elsewhere [10,11], such hierarchy allows one to select singularities in the impulse response that are located on a self-similar set or *support*. This result demonstrates that fractal superlattices, when interrogated by an electromagnetic or optical pulse, imprint their self-similarity properties on the reflected signal in an identifiable way: the support of some detectable singularities in the impulse response is self-similar or fractal. We have shown [11] that the fractal dimension of this support provides an accurate remote estimation of the similarity dimension of the interrogated superlattice. From the hierarchical structures of skeletons we have proceeded to the remote extraction of the descriptors of fractal superlattice such as the stage of growth [10], the similarity dimension *D* (see Fig. 6) [9,11] and the lacunarity ε (see Fig. 7) [12]. We have observed [11] that this approach holds for refractive index differences less than 3 and for temporal standard deviations of pulses less than the inverse of the highest spatial frequency of the interrogated stratified medium. Very recently an efficient wavelet-based partition function has been introduced and successfully applied to the analysis of impulse response of discrete self-similar objects [13].

CONCLUSIONS

Both time-domain and frequency domain techniques provide means for extracting fractal descriptors from stylized one-dimensional fractal objects as exemplified by the Cantor superlattices used here. Work is continuing to expand these results to other cases.

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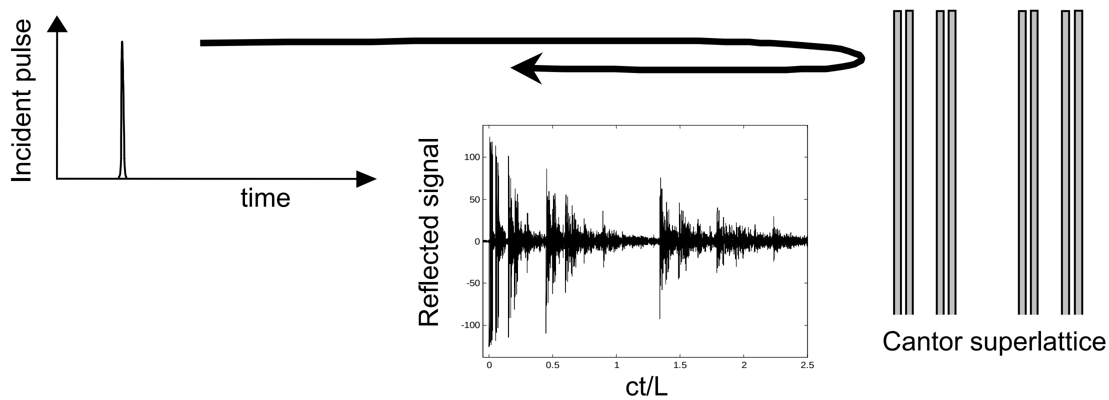

Fig. 3. Electromagnetic or optical Gaussian pulse (top left) incident upon a Cantor superlattice at the seventh stage of growth ($S = 7$) (top right). The reflected signal is wildly irregular (bottom middle).

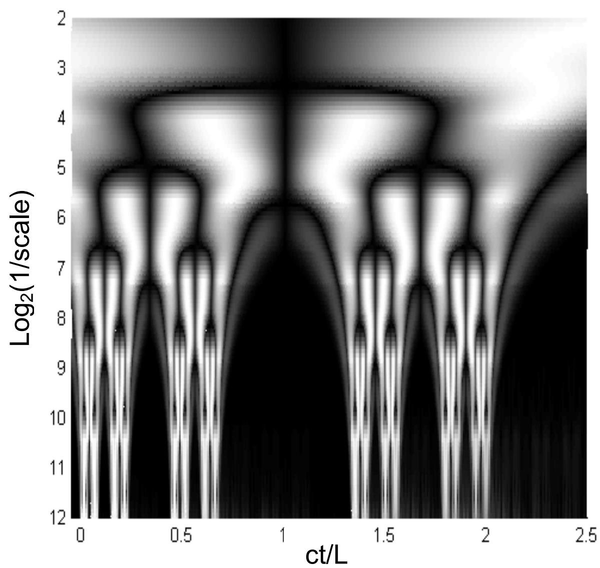

Fig. 4. Wavelet transform modulus of the impulse response of a one-dimensional superlattice (black is zero).

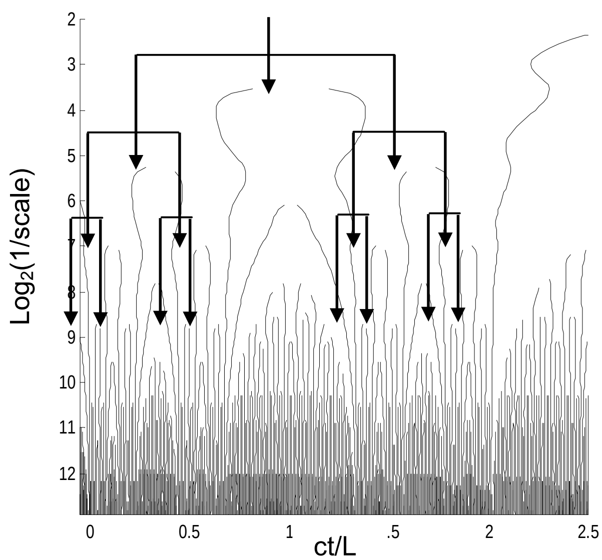

Fig. 5. Wavelet transform modulus maxima of the impulse response of a one-dimensional superlattice. At a given scale, each point indicates the location of a maximum.

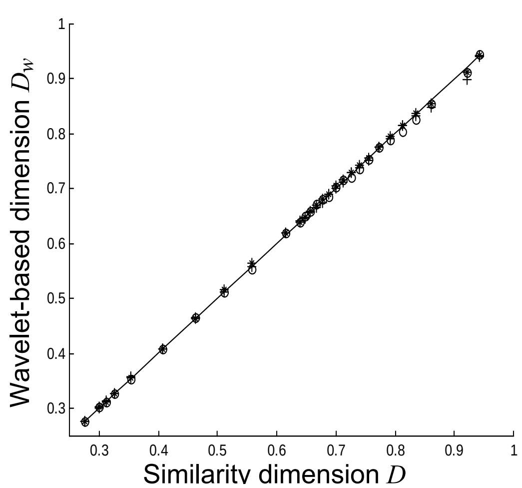

Fig. 6. Wavelet-based dimension D_w as a function of the similarity fractal dimension D for various lacunarity parameters (crosses and circles).

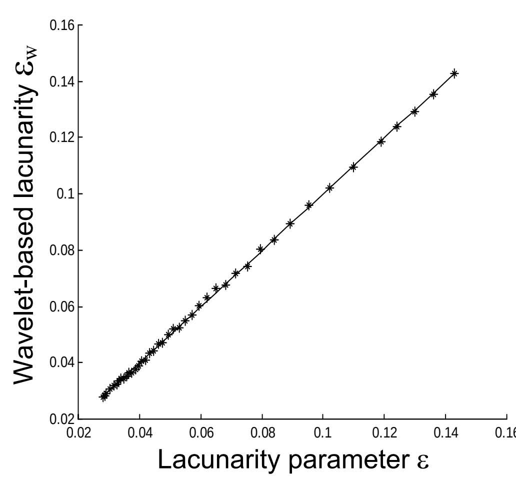

Fig. 7. Wavelet-based lacunarity ε_w as a function of the lacunarity parameter ε for multi-gap superlattices.