NONLINEAR ELECTROMAGNETICS

D. Sjöberg

***Department of Electroscience, Electromagnetic Theory, Lund Institute of Technology,
Box 118 S-221 00 Lund, Sweden, email: daniel.sjoberg@es.lth.se***

1 INTRODUCTION

Linear theories are good theories. This mantra has been repeated many times, and of course contains much truth. Present-day science has well developed methods for dealing with linear problems, where one of the key ingredients is the ability to treat different signals individually and adding the results to obtain the output, that is, using the linearity property. However, there are plenty of phenomena which are not linear, and consequently their models should not be linear either. Examples of these are hysteresis, nonlinear optics, semiconducting devices, shocks, chaotic oscillators etc. This talk aims at giving a tutorial review of some common nonlinear models, and discuss the various problems and possibilities associated with them.

2 REPRESENTATION OF NONLINEAR FUNCTIONALS

We describe three possible representations of a nonlinear constitutive relation: the Volterra expansion, the auxiliary differential equation, and the instantaneous response.

It is often agreed that the most general linear, time invariant constitutive relation is the convolution integral, for instance $\boldsymbol{P} = \chi * \boldsymbol{E}$, where $\boldsymbol{P}$ is the polarization, χ is the susceptibility kernel, $*$ indicates temporal convolution and $\boldsymbol{E}$ is the electric field. The generalization for nonlinear relations is the Volterra series expansion, $\boldsymbol{P} = \chi^{(1)} * \boldsymbol{E} + \chi^{(2)} * \boldsymbol{E} * \boldsymbol{E} + \chi^{(3)} * \boldsymbol{E} * \boldsymbol{E} * \boldsymbol{E} + \cdots$ where each convolution sign $*$ correspond to an integral over a separate time argument, as shown schematically in Fig. 1.

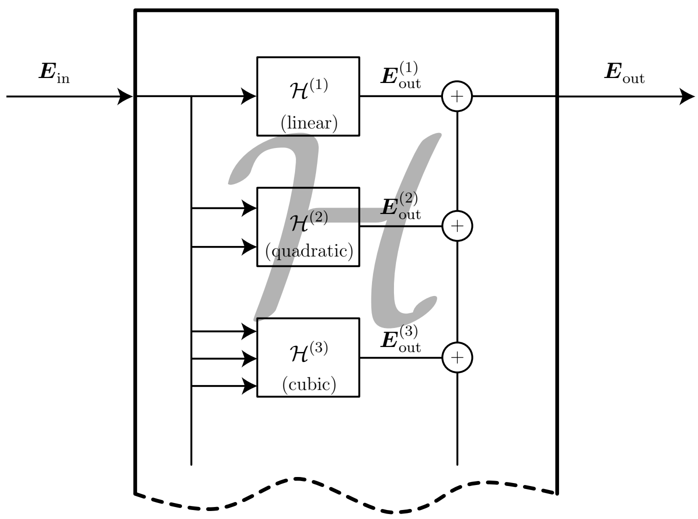

Fig. 1. The Volterra series expansion of a nonlinear map $\mathcal{H}$, operating on a time-dependent field $\boldsymbol{E}_{\text{in}}(t)$. With $\boldsymbol{E}_{\text{out}} = \mathcal{H}\boldsymbol{E}_{\text{in}} = \boldsymbol{E}_{\text{out}}^{(1)} + \boldsymbol{E}_{\text{out}}^{(2)} + \cdots$, each term $\boldsymbol{E}_{\text{out}}^{(n)}(t) = [\mathcal{H}^{(n)}\boldsymbol{E}_{\text{in}}](t)$ is given by the expression $E_{\text{out},i}^{(n)}(t) = \int \cdots \int \chi_{i,p_1\ldots p_n}^{(n)}(t-t_1,\ldots,t-t_n) E_{\text{in},p_1}(t_1)\cdots E_{\text{in},p_n}(t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n$, where summation over indices is implied.

The function $\chi^{(2)}(t_1,t_2)$ has two continuous time arguments, implying that we need to store huge amounts of data in order to use this representation. However, it has some nice theoretical properties, and the data can be greatly reduced by Miller's rule, stating that the higher order susceptibilities often are functions of the linear kernel. This means the second order susceptibility is given by $\chi^{(2)}(t_1,t_2) = a\int \chi^{(1)}(t')\chi^{(1)}(t_1-t')\chi^{(1)}(t_2-t')\,\mathrm{d}t'$, where a is a constant indicating the strength of the nonlinearity, which was discovered experimentally in the sixties [1]. The susceptibility kernels are the Fourier transform of the frequency domain nonlinear susceptibilities used in nonlinear optics [2].

An alternative to the direct representation of the constitutive relation using the Volterra series, is to define the material response as the solution to a nonlinear differential equation. The most familiar example is probably the Landau-Lifshitz equation describing the magnetization in a ferromagnetic material: $\boldsymbol{M}'(t) = g\boldsymbol{M}(t)\times\boldsymbol{H}(t) - \lambda\boldsymbol{M}(t)\times(\boldsymbol{M}(t)\times\boldsymbol{H}(t))$, where $\boldsymbol{M}$ is the magnetization, $\boldsymbol{H}$ is the magnetic field strength and g and λ are constants [3]. Various generalizations of the Lorentz and Debye differential equations describing the time evolution of the polarization $\boldsymbol{P}$ are also possible. Such differential equations can often be deduced from physical reasoning, and thus have a direct connection to the physics of the problem. However, they may also contain very large/small parameters, which may render them numerically intractable and require some suitable renormalization.

The two previous approaches deal directly with the fact that most materials have some sort of memory effects, that is, dispersion. Sometimes, the signals we are interested in are much slower than the lowest resonance frequency of the material, which justifies looking only at an instantaneous response. The constitutive functional then takes the form of a function, that is, for each point $(\boldsymbol{x}, t)$ in space and time we have $\boldsymbol{P}(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{E}(\boldsymbol{x}, t))$, where $\boldsymbol{f}$ may be a nonlinear function. When this relation is combined with the Maxwell equations, the result is a quasi-linear system of partial differential equations, which usually support shock solutions, that is, solutions which become discontinuous in finite time even if the initial data is smooth.

3 WELL-POSEDNESS, BLOW-UP

Using a nonlinear constitutive functional in combination with the Maxwell equations often results in a system of nonlinear partial differential equations. The mathematical knowledge of these systems is limited (at least compared to linear equations), meaning there is often no guarantee that we have unique solutions or even that solutions exist in the classical sense. A rule of thumb is that if the principal part of the equation (the highest order derivatives) is linear, it is often possible to lean back on linear results, at least for small time intervals. Another general remark is that much more is known about one spatial dimension than three.

One common phenomenon in nonlinear partial differential equations, is that smooth solutions are no longer available, even if the geometry of the domain and the initial/boundary values are smooth. This necessitates the search of generalized solutions, which may be discontinuous or only defined as a probability measure [4]. One problem with such solutions is the numerical treatment, which is usually based on assumptions such as continuity. Even if the solution can be found explicitly, it may blow up in finite time, as the simple example $u'(t) = u(t)^2$, $u(0) = 1$, with the solution $u = 1/(1-t)$, demonstrates. Techniques to extract valuable information from ill-behaved nonlinear models are for instance singular perturbations and multiple scale expansions [5, 6]. These methods lead to new nonlinear models, in the hope that the new models are well posed. Physical examples of nonlinear blow-up is the self-focusing of a laser beam (strong enough to partially melt crystals), and the formation of shock waves [7].

Fig. 2. A pulse propagating through a nonlinear medium is sharpened on its trailing edge. This can be described as a breakdown of the gradient of the field, and generates lots of higher order harmonics in a frequency domain description.

4 COMPLEXITY OF FREQUENCY INTERACTION

It is easily seen that the different frequency components of an electromagnetic field modeled by nonlinear equations must couple to each other. For weak nonlinearities and a fixed input frequency, this takes the form of the generation of harmonics. In nonlinear optics, this is often described as the absorption of two photons with equal frequency, and the emission of one photon with the double frequency, two-photon absorption. This requires that the sum of the original photon energies correspond to an energy gap in the atom, which obviously involves the dispersive properties of the medium. It can be shown that an essential macroscopic condition for the generation of a new electromagnetic wave with frequency ω and wave vector $\boldsymbol{k}$, is that the frequencies and wave vectors of the two waves combined to create this wave satisfy the relations $\omega_1 + \omega_2 = \omega$ and $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}$. We must also require that $\omega_1 = \omega(\boldsymbol{k}_1)$, $\omega_2 = \omega(\boldsymbol{k}_2)$ etc, where $\omega(\boldsymbol{k})$ is the dispersion relation corresponding to the linear part of the model. This is called phase matching, and is often accomplished with the help of anisotropic crystals, where the refractive index depends both on the direction and the magnitude of the wave vector.

Using high-intensity lasers enables the generation of high-order harmonics. In some experiments harmonics of orders exceeding several hundreds are measured, and the sheer complexity of the frequency interaction is a formidable task, much similar to the case of multiple scattering in scattering theory. One tool used to keep track of all the combinatorial possibilities is Feynman diagrams [2].

5 HYSTERESIS AND PHASE TRANSITIONS

One of the most familiar examples of nonlinearity is the hysteresis curve associated with the magnetization of a ferromagnetic material. It can be linearized at each point, but the linearization breaks down as soon as we apply large variations in the magnetic field. Hysteresis is a phenomenon which typically arises in the vicinity of phase transitions in a material, and the hysteresis loop is a manifestation of the fact that it takes some energy to push the system from one phase to another. This is why ferroelectrics have a strongly nonlinear permittivity. Phase transitions are often sensitive to temperature changes. For instance, ferromagnetism breaks down when the temperature exceeds the Curie temperature.

The Landau-Lifshitz equation mentioned earlier can be used to model the flipping of magnetic domains in a ferromagnet, and in recent years there has appeared a more general mathematical notion of hysteresis operators. In particular, the connection between hysteretic behavior and partial differential equations, describing wave propagation or diffusion in a medium with hysteresis, has been investigated [8, 9].

6 SHOCK STRUCTURE

We have previously mentioned that solutions to nonlinear partial differential equations may not be continuous. One manifestation of this is the spontaneous formation of shocks from smooth inital data, that is, the spontaneous appearance of discontinuous waves. However, it turns out that not all discontinuities are admissible as to the second law of thermodynamics; some discontinuities increase the entropy, and others do not. One mathematical description of this is the shock structure problem, where stable travelling waves $\boldsymbol{E}(x - vt)$ are sought. The techniques comprise methods from anisotropic wave propagation, singular perturbation, and dynamical systems [10, 11].

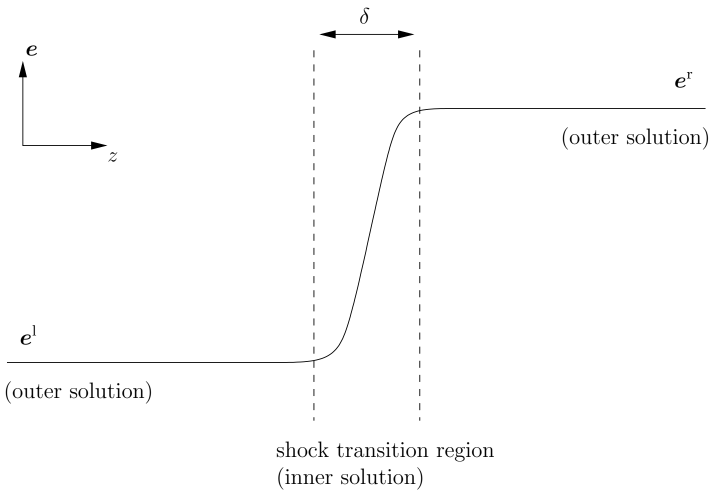

Fig. 3. The shock structure problem. The discontinuous wave is considered as a limit of continuous waves, where the transition between the outer (constant) solutions $\boldsymbol{e}^{\mathrm{l}}$ and $\boldsymbol{e}^{\mathrm{r}}$ occurs in a transition layer of width δ. Physical solutions correspond to a dissipation of electromagnetic energy within the transition layer.

7 DISCUSSION

We have mentioned some problems in the modeling of nonlinear electromagnetics, and indicated a few of the methods available to deal with them. Among the fields which have a stronger tradition of dealing with nonlinear models, we mention fluid and gas dynamics, magnetohydrodynamics, and micromagnetics. There is much to expect from the mathematical and numerical understanding of nonlinearities which has evolved during the last decades, but not yet become commonplace in the applied sciences.

REFERENCES

[1] R. C. Miller. "Optical second harmonic generation in piezoelectric crystals (ferroelectrics; KH_2PO_4, KD_2PO_4, $NH_4H_2PO_4$, KH_2AsO_4, quartz, $AlPO_4$, ZnO, CdS, GaP, GaAs, $BaTiO_3$; T/E)". *Appl. Phys. Lett.*, vol. 5(1), pp. 17–19, Jul. 1964.

[2] R. W. Boyd. *Nonlinear Optics*. San Diego: Academic Press, 1992.

[3] L. D. Landau and E. M. Lifshitz. "On the theory of the dispersion of magnetic permeability in ferromagnetic bodies". *Physik. Z. Sowjetunion*, vol. 8, pp. 153–169, 1935. Reprinted by Gordon and Breach, Science Publishers, "Collected Papers of L. D. Landau", D. Ter Haar, editor, 1965, pp. 101–114.

[4] M. Taylor. *Partial Differential Equations III: Nonlinear Equations*. New York: Springer-Verlag, 1996.

[5] J. Kevorkian and J. D. Cole. *Multiple Scale and Singular Perturbation Methods*. Berlin: Springer-Verlag, 1996.

[6] C. Sulem and P.-L. Sulem. *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*. Berlin: Springer-Verlag, 1999.

[7] G. Branch and P. W. Smith. "Fast-rise-time electromagnetic shock waves in nonlinear, ceramic dielectrics". *J. Phys. D: Applied Phys.*, vol. 29(8), pp. 2170–2178, Aug. 1996.

[8] A. Visintin. *Differential Models of Hysteresis*. Berlin: Springer-Verlag, 1994.

[9] M. Brokate and J. Sprekels. *Hysteresis and Phase Transitions*. Berlin: Springer-Verlag, 1996.

[10] D. Sjöberg. "Shock structure for electromagnetic waves in bianisotropic, nonlinear materials". *Radio Science*, 2002. Accepted for publication.

[11] J. Smoller. *Shock Waves and Reaction–Diffusion Equations*. Grundlehren der mathematischen Wissenschaften 258. Berlin: Springer-Verlag, 2nd edn., 1994.