ON FOSTER'S REACTANCE THEOREM FOR METAMATERIALS WITH NEGATIVE PERMITTIVITY AND PERMEABILITY

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ABSTRACT

In this work, it is shown that Foster's reactance theorem is satisfied for a one-port termination filled with lossless metamaterials with negative real permittivity and permeability. However, when the reactive input impedance of such a termination is compared with that of its counterpart filled with conventional lossless materials, it is found that the two reactances may have opposite signs. So if a one-port termination filled with lossless dielectric with positive permittivity and permeability possesses inductive input reactance, when the same termination is filled instead with a lossless material with negative real permittivity and permeability, its input reactance may be capacitive.

STATEMENT OF THE PROBLEM

Composite materials with negative permittivity *and* permeability at some frequencies have recently attained considerable attention and interest [1]-[11]. This original idea goes back to 1967 when Veselago theoretically studied time-harmonic monochromatic plane wave propagation in a material whose permittivity and permeability were assumed to be simultaneously negative [6]. Shelby, Smith, and Schultz recently constructed such a composite medium for the microwave regime, and experimentally showed the presence of anomalous refraction in this medium [4]. It is also interesting to note that previous theoretical study of electromagnetic wave interaction with omega media using the circuit-model approach had also conceptually revealed the possibility of having negative permittivity and permeability in omega media for certain range of frequencies [7].

Recently as a potential application of these metamaterials I introduced theoretically an idea for compact one-dimensional (1-D) cavity resonators, in which a combination of a slab of conventional material and a slab of metamaterial with negative permittivity and permeability was inserted [11]. An interesting issue was raised by Munk about the applicability of Foster's reactance theorem for the metamaterials with negative permittivity and permeability [12]. As is well known, if one has a one-port reactive lossless termination in a microwave network, the input impedance Z_{in} of such a one-port termination is purely imaginary, i.e., $Z_{in} = jX_{in}$, (and of course $Y_{in} = jB_{in}$). So the input impedance (and input admittance) is purely reactive. (Here the time dependence of $\exp(j\omega t)$ is assumed.) Foster's reactance theorem states that for such a one-port reactive termination [13] , we have

$$\frac{\partial X_{in}}{\partial \omega} > 0 \qquad (1)$$

$$\frac{\partial B_{in}}{\partial \omega} > 0 \qquad (2)$$

This implies that the poles and zeros of a reactance (or a susceptance) function must alternate along the frequency axis [13]. The important issue is to investigate how the metamaterials with negative permittivity and permeability satisfy Foster's Reactance Theorem.

Consider a general cavity with perfectly conducting wall filled with a lossless metamaterial with $\mu < 0$ and $\varepsilon < 0$ at a certain band of frequency, and assume an open port at the wall of this cavity. The open port is connected to a cylindrical waveguide having a cross section similar to the shape of the open port. This waveguide has also perfectly conducting walls and is filled with the same metamaterial as in the cavity. (See Fig. 1) This would form a one-port lossless termination, in which such a metamaterial is inserted. If we follow the mathematical steps of Foster's reactance theorem described in [13], we obtain

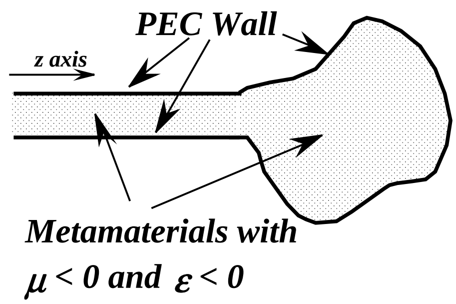

Fig. 1. Geometry of the problem.

$$\int_{\text{terminal plane}} \left(\boldsymbol{E} \times \frac{\partial \boldsymbol{H}^*}{\partial \omega} + \frac{\partial \boldsymbol{E}^*}{\partial \omega} \times \boldsymbol{H} \right) \cdot \boldsymbol{n}\, dS = -j \int_V \left(\boldsymbol{H} \cdot \boldsymbol{H}^* \frac{\partial(\omega\mu)}{\partial \omega} + \boldsymbol{E} \cdot \boldsymbol{E}^* \frac{\partial(\omega\varepsilon)}{\partial \omega} \right) dV \tag{3}$$

where the "terminal plane" is any transverse plane in the waveguide at some $z = z_o$ along the z axis (this is the "port" to this one-port termination.), *V* is the volume inside the cavity and including the part of the waveguide up to the terminal plane, ***n*** is a unit normal, and the superscript (*) indicates complex conjugation. This result does not depend on the sign of ε and μ. We assume that the cross sectional size of the waveguide is chosen such that only the dominant mode of the waveguide can propagate. Therefore, if the terminal plane is far enough away from the opening at the cavity wall, the dominant mode of the waveguide is the only mode present at the terminal plane, and this mode forms the standing wave in this waveguide. At the terminal plane, we define the transverse input impedance Z_{in} as

$$Z_{in} \boldsymbol{H}_{trans} = \hat{\boldsymbol{z}} \times \boldsymbol{E}_{trans} \tag{4}$$

with $\boldsymbol{E}_{trans}$ and $\boldsymbol{H}_{trans}$ being the transfer components of the electric and magnetic fields at the terminal plane, and $\hat{\boldsymbol{z}}$ being the unit vector along the waveguide axis, i.e., z axis. Since this one-port termination is assumed to be lossless, the transverse input impedance is purely imaginary, i.e., $Z_{in} = jX_{in}$. If we continue the mathematical steps from (3) we find, similar to the proof of Foster's reactance theorem given in [13], that

$$\text{sign of } \frac{\partial X_{in}}{\partial \omega} = \text{ sign of } \left(W_m + W_e \right) \tag{5}$$

where W_m and W_e are the time-average energy stored in the lossless termination volume *V*, and are given as

$$W_m = \tfrac{1}{4} \int_V \boldsymbol{H} \cdot \boldsymbol{H}^* \frac{\partial(\omega\mu)}{\partial \omega} dV \tag{6}$$

$$W_e = \tfrac{1}{4} \int_V \boldsymbol{E} \cdot \boldsymbol{E}^* \frac{\partial(\omega\varepsilon)}{\partial \omega} dV \tag{7}$$

It is important to note that metamaterials with negative permittivity and permeability are inherently quite dispersive. So although ε and μ can be negative at a certain band of frequencies in such media, these parameters do vary with frequency quite noticeably. In fact, $\frac{\partial(\omega\mu)}{\partial\omega}$ and $\frac{\partial(\omega\varepsilon)}{\partial\omega}$ should be positive in order to guarantee the time-average stored electric and magnetic energy densities to be positive quantities. Therefore, we find that

$$\frac{\partial X_{in}}{\partial \omega} > 0 \tag{8}$$

Foster's reactance theorem is indeed satisfied for these materials, similar to the case of conventional materials with real positive permittivity and permeability. However, an important question to ask is the following: Although $\frac{\partial X_{in}}{\partial\omega} > 0$ for these materials, what is the sign of $\frac{\partial X_{in}}{\partial z}$ for a one-port termination filled with these materials with negative ε and μ? This is an important distinction one has to make between a conventional material and this material with negative permittivity and permeability. To answer this question, without loss of generality we assume that the dominant mode of the waveguide is a TE mode. (A similar proof can be given for the case of a TM mode.) For this dominant mode, the transfer component of the magnetic field at the terminal plane can be written as

$$\boldsymbol{H}_{trans} = \tfrac{1}{-j\omega\mu}\hat{\boldsymbol{z}} \times \frac{\partial \boldsymbol{E}_{trans}}{\partial z} \tag{9}$$

If we substitute the above relation into (4) and noting that $Z_{in} = jX_{in}$, we get

$$X_{in}\,\hat{\boldsymbol{z}} \times \frac{\partial \boldsymbol{E}_{trans}}{\partial z} = -\omega\mu\,\hat{\boldsymbol{z}} \times \boldsymbol{E}_{trans} \tag{10}$$

Since we have a standing wave for the dominant mode in the waveguide at the terminal plane, the transverse electric field components can be written in the general form as

$$\boldsymbol{E}_{trans} = \boldsymbol{f}(x,y)\sin(\beta z + \phi) \tag{11}$$

where $\boldsymbol{f}(x,y)$ is a transverse vector in the x-y plane, as a function of x and y coordinates, β is the longitudinal wavenumber for the dominant TE mode, and ϕ is an arbitrary phase. Substituting (11) into (10), we get

$$X_{in}\,\beta\hat{\boldsymbol{z}} \times \boldsymbol{f}(x,y)\cos(\beta z + \phi) = -\omega\mu\,\hat{\boldsymbol{z}} \times \boldsymbol{f}(x,y)\sin(\beta z + \phi) \tag{12}$$

From this equation, we notice that when we compare the case of a conventional medium with positive μ and ε with that of a medium with negative μ and ε, the sign of X_{in} would flip, regardless of what sign we choose for β. So if at the terminal plane of the one-port termination filled with conventional lossless dielectric with $\mu_1 > 0$ and $\varepsilon_1 > 0$, we have an inductive reactance, when we exchange the filling medium with a lossless material with $\mu_2 = -\mu_1$ and $\varepsilon_2 = -\varepsilon_1$, the input reactance at the same terminal plane will be capacitive. This implies that whatever the sign of $\frac{\partial X_{in}}{\partial z}$ is for the former, $\frac{\partial X_{in}}{\partial z}$ has an opposite sign for the latter. So although Foster's reactance theorem is satisfied for metamaterials with negative permittivity and permeability, as is for the conventional materials, the rate of change of reactance X_{in} with respect to the location of the terminal plane, i.e., $\frac{\partial X_{in}}{\partial z}$, differs for the case of metamaterials with negative permittivity and permeability.

ACKNOWLEDGEMENT

I like to thank Professor Ben A. Munk of the Ohio State University for raising the issue of Foster's reactance theorem during our discussion at the *International Conference on Electromagnetics in Advanced Applications (ICEAA'01),* in Torino, Italy, September 10-14, 2001.

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