SCATTERING ON ROUGH SURFACES WITH ALPHA-STABLE NON-GAUSSIAN HEIGHT DISTRIBUTIONS

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ABSTRACT

We study the electromagnetic scattering problem on a random rough surface when the height distribution of the profile belongs to the family of alpha-stable laws. This allows us to model peaks of very large amplitude that are not accounted for by the classical Gaussian scheme. For such probability distributions with infinite variance the usual roughness parameters such as the RMS height, the correlation length or the correlation function are irrelevant. We show, however, that these notions can be extended to the alpha-stable case and introduce a set of adapted roughness parameters that coincide with the classical quantities in the Gaussian case. Then we study the scattering problem on a stationary alpha-stable surface and compute the scattering coefficient under the first-order Kirchhoff and Small-slope approximation. An analytical formula is given in the high-frequency limit, which generalizes the well-known Geometrical Optics approximation.

INTRODUCTION

In the theory of scattering by random rough surfaces, a systematic yet crucial assumption is the Gaussianity of the height distribution. This hypothesis is adopted for technical reasons but is often far from being realistic. Gaussian distributions miss essentially two important features of natural surfaces. First, they are unskewed, that is unable to explain an eventual asymmetry between crests and troughs. Second, the tail of these distributions is exponentially decreasing, thereby not allowing the occurrence of “big” events; Gaussian distributions are therefore also irrelevant for rough surfaces that can exhibit peaks of very large amplitude with respect to the mean level. In this paper, we will focus on the second aspect only and discard for this first attempt the modelization of skewness. Surfaces with a non-negligible probability of very large peaks are well described by heavy-tailed height distributions, that is distributions whose tail is polynomially (instead of exponentially) decreasing. The archetype of such distributions are the stable distributions. This family of distributions share with the Gaussian distribution the essential property of being stable under linear combination of independent variables and also a Generalized Central Limit Theorem for normalized sums of independent identically distributed variables. This is why they are found in the descriptions of many physical and economical systems. To our best knowledge, very few works have been concerned with non-Gaussian surface scattering [1] [2][3] [4] [5]. We take another step in direction of non-Gaussian surfaces by studying the family of alpha-stable surfaces. Contrarily to the previous models, they are not second-order (that is they have infinite variance) and thus require a complete redefinition of the roughness parameters. We will show that, in spite of their apparent complexity, alpha-stable surfaces yield to simple expression for the scattering computations. We have chosen to work in the one-dimensional case, which means that the surface is assumed to be invariant in one direction and hence only described by its profile. This restriction has been adopted to make the presentation clearer in introducing new concepts and to avoid some technical difficulties. The extension to the two-dimensional case is, however, straightforward.

The SCATTERING PROBLEM

A z-invariant infinite surface $y = h(x)$ separates the vacuum (upper medium) from a perfectly conducting medium. A linearly polarized time-harmonic plane wave with wave vector $\mathbf{K_0} = (k_0, -q_0)$ and wave-number $K = \sqrt{k_0^2 + q_0^2} = 2\pi/\lambda$ is impinging at incidence θ_0 on the top of the surface. The total field F in the upper medium writes $F = F_i + F_s$, where F_i is the incident field, $F_i(x,y) = q_0^{-1/2}\exp(i\mathbf{K_0}\cdot\mathbf{r}) = q_0^{-1/2}\exp(ik_0x - iq_0y)$, and F_s is the scattered field. Here F stands for the electric field E or magnetic field H according to whether the polarization is *s* or *p*. It satisfies the Helmholtz equation in the upper domain ($\Delta F + K^2F = 0$) and vanishes in the lower domain ($F = 0,\ y < f(x)$) with Dirichlet (resp. Neumann) boundary conditions on the interface in the *s* (resp. *p*) polarization case. The harmonic time-dependence $e^{-i\omega t}$ will always be implicit. Above the highest excursion of the surface, the scattered field admits a Rayleigh expansion:

$$F_s(x,y) = \int_{\mathbb{R}} S(k,k_0)q^{-1/2}\exp(i\mathbf{K}\cdot\mathbf{r})dk = \int_{\mathbb{R}} S(k,k_0)q^{-1/2}\exp(ikx + iqy)dk$$

Here and everywhere, k and q stand for the horizontal and vertical component of the wave vector $\mathbf{K}$, $k = K\sin\theta,\ q = \sqrt{K^2 - k^2}$, where the square root defining q is taken to be positive imaginary when $k > K$ (this corresponding to evanescent waves). The complex coefficient $S(k,k_0)$ is called scattering amplitude. There is in general no closed analytical formula for the latter, which has to be evaluated numerically. However, under some physical assumptions, the scattering amplitude admits simple analytical approximations. We will rely on two widely used approximations, namely the first-order Kirchhoff Approximation (KA) and Small-Slope Approximation (SSA) introduced recently by Voronovich [6],[7]. The former is known to be valid when the surface is smooth at the resolution of the electromagnetic wavelength, roughly speaking, while the latter becomes accurate when the mean surface slope goes to zero. In the KA as well as SSA, the scattering amplitude can be written:

$$S(k,k_0) = \frac{2(qq_0)^{1/2}}{q+q_0}B(k,k_0)\frac{1}{2\pi}\int_{\mathbb{R}}\exp[-i(k-k_0)x - i(q+q_0)h(x)]dx$$

where the optical factor $B(k,k_0)$ depends only on the geometry, the polarization and the considered approximation. The measurable quantity of interest is the scattering coefficient $I(k,k_0) =< |S(k,k_0)|^2 >$, where $< >$ denotes the ensemble average.

SURFACE MODEL

A random variable X is said to be symmetric alpha-stable (written $X \sim S_\alpha(\gamma,\beta,\mu)$) with scale parameter γ if its characteristic function satisfies $< \exp(iuX) >= \exp(-|u|^\alpha \gamma^\alpha)$. Hence a $S\alpha S$ variable has a probability distribution function:

$$p_\alpha(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}\exp(-|u|^\alpha)du. \tag{1}$$

Note that a S_2S variable with scale parameter γ corresponds to a Gaussian variable $N(0,2\gamma^2)$. There are no closed analytical formulae for the probability distribution functions of stable variables, except in the particular cases $\alpha = 2$ (Gaussian distribution), $\alpha = 1$ (Cauchy distribution) and $\alpha = 1/2$ (Levy distribution). However for $0 < \alpha < 2$ the following asymptotic behavior holds,

$$p_\alpha(x) \simeq \alpha c_\alpha x^{-\alpha-1},\ x \to +\infty,$$

with $c_\alpha = \frac{1}{\pi}\Gamma(\alpha)\sin(\alpha\pi/2)$. Hence stable distributions are heavy-tailed with a power-law decay related to their stability index. An essential difference between non-Gaussian stable and Gaussian variables is the occurrence of infinite moments for the former, as can be seen from the previous formula. In particular stable variables have infinite variance for $\alpha < 2$, and infinite absolute mean for $\alpha < 1$. We will assume $1 < \alpha \leq 2$ for technical reasons. A stable process is a process whose finite multi-dimensional distributions are stable (we refer to [8] for the definition of stable vectors since this would bring us too far off the subject). Unlike Gaussian processes, a stable process cannot be determined by its two-points distribution function only. However, one can take advantage of the stability property to construct stable processes by linear summation of stable noise. For this, recall the definition of the $S_\alpha S$ Levy motion, which is the stable analogue of Brownian motion. This is a process $M(x)$ with independent increments such that $M(x) - M(y) \sim S_\alpha(|x-y|^{1/\alpha}, 0, 0)$ and $M(0) = 0$ almost surely. Then any process of the form

$$h(x) = \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} F(x-y) dM(y), \tag{2}$$

where F is some measurable function with $\int |F|^\alpha < \infty$, is a stationary $S_\alpha S$ process, with scale parameter $\gamma = \frac{1}{\sqrt{2}} \left(\int |F(x)|^\alpha \right)^{1/\alpha} := \frac{1}{\sqrt{2}} \|F\|_\alpha$. The stochastic integral (2) is well defined when F is piece-wise constant. In the general case, it is defined by approximating F by such functions and taking the limit in probability. The $1/\sqrt{2}$ factor in the MA representation has been set so as to coincide with the Gaussian case for $\alpha = 2$, since it that case $2^{-1/2} dM(x) = dB(x)$ corresponds to standard Brownian motion.

The usual RMS height is no longer relevant for stable non-Gaussian surfaces since the latter have infinite variance. However, a natural extension of the RMS height is the (renormalized) scale parameter $\sigma = \sqrt{2}\gamma = \|F\|_\alpha$. Note that $X \sim S_\alpha(\sqrt{2}\sigma, 0, 0) \Rightarrow aX \sim S_\alpha(a\sqrt{2}\sigma, 0, 0)$, so that σ is actually a good indicator of the vertical scale of the process. Likewise, the correlation function is not defined for $\alpha < 2$ but can be replaced by its stable analog for $1 < \alpha \leq 2$, the so-called covariation function, which in the case of a MA process (2) can be defined as:

$$C_\alpha(r) = \int_{\mathbb{R}} dx F(x) F^{<\alpha-1>}(x+r).$$

Here $a^{<p>}$ is the signed power of a, that is $a^{<p>} = \text{sign}\,(a)\,|a|^p$. The covariation has in general weaker properties than the covariance; in particular a vanishing covariation does not necessarily imply independence of the corresponding variables. However, in the case of a MA representation (2) with nonnegative filter, note that $C_\alpha(r) = 0$ implies $F(x)F^{<\alpha-1>}(x+r) = 0$ a.e, which in turn implies the independence of $h(x)$ and $h(x+r)$ (see e.g. [8] p 129). It can be shown that $\sigma^\alpha = C_\alpha(0)$, $C_\alpha(r) \leq \sigma^\alpha$ and $C_\alpha(r) \to 0$ as $r \to \infty$. Hence, the covariation function is a good indicator of the dependence of points with respect to their mutual distance. A natural extension of the correlation length is then the shortest distance l for which $C_\alpha(l) = C_\alpha(0)\exp(-1)$. As in the Gaussian case, the differentiability properties of stable processes depend on the regularity of the filter. For $1 < \alpha \leq 2$, the sample paths are almost surely differentiable as soon as F has a derivative F' in $L^\alpha(\mathbb{R})$. The differentiated process h' is then again $S_\alpha S$ process with scale parameter $\|F'\|_\alpha$. By analogy with the Gaussian case, we will identify the quantity $\|F'\|_\alpha$ with the slope parameter. The quantities $C(r), \sigma$ and l coincide with the correlation function, the RMS heights and the correlation length, respectively, in the Gaussian case $\alpha = 2$.

THE SCATTERING COEFFICIENT FOR STABLE SURFACES

We will now assume that the surface profile $h(x)$ is a $S_\alpha S$ process with $1 < \alpha \leq 2$ and compute the corresponding scattering coefficient. The process will be given by the MA representation (2) with filter $F_{\sigma,l}$ (scale parameter σ, correlation length l). We obtain the following expression for the scattering coefficient:

$$I(k,k_0) = |B(k,k_0)|^2 \frac{1}{2\pi} \frac{4qq_0}{(q+q_0)^2} \int_{\mathbb{R}} dx e^{-i(k-k_0)x} \left(\exp[-2^{-\alpha/2}(q+q_0)^\alpha \sigma^\alpha_{cross}(x)] - \exp[-2^{1-\alpha/2}(q+q_0)^\alpha \sigma^\alpha] \right),$$

where $\sigma^\alpha_{cross}(x) = \int_{\mathbb{R}} du \left| F_{\sigma,l}(x+u) - F_{\sigma,l}(u) \right|^\alpha$. For large Rayleigh parameter $q_0\sigma$ we have the approximate formula for the scattering coefficient:

$$I(k,k_0) = \frac{4qq_0}{(q+q_0)^3} \frac{\sqrt{2}}{s} |B(k,k_0)|^2 p_\alpha \left(\frac{\sqrt{2}(k-k_0)}{s(q+q_0)} \right), \tag{3}$$

where p_α is the $S_\alpha S$ probability distribution function (1) and $s = \sigma l^{-1} \|F'\|_\alpha$ is the slope parameter. For $\alpha = 2$ we recover the well-known formula (e.g. [9] p127):

$$I(k, k_0) = \frac{4qq_0}{s(q+q_0)^3} |B(k, k_0)|^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(k-k_0)^2}{2s^2(q+q_0)^2}\right),$$

which is often referred to as the Geometrical Optics (GO) approximation of the scattering coefficient, as it is corresponds to the high-frequency limit. Note that in the GO approximation (3) the scattering coefficient depends only on the slope parameter s, as in the Gaussian case. Furthermore the scattering diagram (i.e the intensity I as a function of the scattering angle θ, for fixed incidence θ_0) has the following remarkable property: its shape is given by the height distribution function p_α (providing the change of representation $\theta \to (k-k_0)/(q+q_0)$). This can be interesting for an estimation procedure in the inverse problem (that is to estimate α from the scattering coefficient, or to determine whether the surface is alpha-stable or not) but will not be discussed here.

CONCLUSION

In this paper we have have studied the scattering problem on a one-dimensional rough surface modeled by a stationary stable process. We have shown that the usual roughness parameters that are used for Gaussian surfaces can be extended to the larger family of stable surfaces and are a particular case thereof. Finally, we have derived analytical approximations for the scattering coefficient in the framework of first-order approximations (KA and SSA). This is only the beginning of a fruitful field of investigation: important questions remain such as to find an estimation procedure for the stability index in the inverse problem. Also, this work has to be extended to the two-dimensional case and the model to be confronted with real data.

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