A STEP-WISE APPROACH TO INVERSE SCATTERING

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ABSTRACT

In this paper we introduce a step-wise refinement procedure for the solution of inverse scattering problems. The basic idea is that of splitting the problem into several parts, each devoted to gain a part of the solution of the whole problem. By so doing, one gradually achieves an increased amount of information on the surveyed objects, while at the same time relaxing the scope of each problem. In particular, a two-steps procedure is presented and discussed wherein the first step aims to find the support of the object, whereas the second step exploit this information in order to get a quantitative reconstruction of the unknown profile.

1. INTRODUCTION AND RATIONALE

In the recent years, solution of inverse scattering problems is becoming a topic of potential interest in a large number of applications. However, these problems are usually non-linear and ill-posed, so that it is very important to develop approaches which are able to guarantee accurate and reliable solutions. Both ill-posedness and non-linearity suggest to look for a finite dimensional representation of the unknown with a number of terms as low as possible [1]. Unfortunately, the "optimal" choice of kind and dimensionality of the representation is generally unknown.

It is therefore of the outmost importance the fact that some properties of the scatterer, such as its convex envelope [2] or even its (possibly) non convex support [3] can be retrieved without explicitely solving the inverse problem. This circumstance suggests to split the problem into several parts. In each of them, one aims to solve a simple problem such to gain a part of the solution of whole problem. By so doing, one gradually achieves an increased amount of information on the surveyed objects, while at the same time relaxing the scope of each problem.

Accordingly, we propose in this paper a novel two-step strategy. The first step is aimed to retrieve the possibly non-convex support of the scatterer *without* explicitly solving the inverse problem. The second step consists of the quantitative reconstruction of the permittivity, which is pursued exploiting the information achieved in the previous step. This step is based on an inversion approach which can reduce "false solution" occurrence by means of "synthetic" zooming properties of wavelets (see Section 3).

The goal of the first step is achieved exploiting and re-thinking the "simple method" proposed in [3]. Let us therefore briefly recall these results.

Let $\mu_\infty(\theta,\varphi)$ be the field scattered in the direction φ due to a unitary plane wave impinging from θ, and let y_0 be a generic point of the investigated region Ω containing a scatterer with support D. For each y_0 in Ω, consider the integral equation in the unknown $g(y_0,\theta)$:

$$\int_{-\pi}^{\pi}\mu_\infty(\theta,\varphi)g(y_0,\theta)d\theta = \exp[+jk\rho\cos(\varphi-\alpha)] \tag{1}$$

wherein $y_0 = (\rho\cos\alpha, \rho\sin\alpha)$. Then, if $\varepsilon > 0$, there exists a function $g \in L^2$ such that

$$\left\| \int_{-\pi}^{\pi}\mu_\infty(\theta,\varphi)g(y_0,\theta)d\theta - \exp[+jk\rho\cos(\varphi-\alpha)] \right\| < \varepsilon \tag{2}$$

and such that if

$$y_0 \to \partial D \quad (y_0 \in D) \text{ then } \|g(y_0,\cdot)\| \to \infty \tag{3}$$

that is, the generalized solution of (1) becomes unbounded as y_0 approaches (from the inside) the boundary of the scatterer. Let us give a physical interpretation of such a result in case of dielectric scatterers. As the right hand member in (1) is the far field pattern of a cylindrical wave emerging from , y_0 problem (1) can be stated as:

Find $g(y_0,\theta)$ such to combine the scattered fields so that a cylindrical wave appears to emerge from y_0.

This could be physically realized by combining the incident fields according to $g(y_0,\theta)$ and it is somehow equivalent to focus the radiating component of the induced currents in y_0. This consideration leads to a simple interpretation of (3). In fact, while it is possible to focus the induced current in y_0 as long as it belongs to *D*, this is not any more possible when $y_0 \notin D$. In such a case, even assuming of being able to focus the total field in y_0, the induced current would be zero and, in order to compensate for such a null, $\| g \|$ becomes unbounded in these points. Therefore, inversion of (1) for each y_0, together with (3), would allow to estimate the unknown shape of the scatterer(s) in a straightforward fashion. However, this means solving a first kind Fredholm equation involving a compact operator, which is an ill-posed problem.

In the following, by relying on an alternative formulation of the inverse scattering problem introduced in the next Section, we reformulate the simple method in a regularized framework (Section 3).

2. AN ALTERNATIVE FORMULATION OF THE INVERSE SCATTERING PROBLEM

By the sake of simplicity, we will refer throughout to the 2D scalar case, so that the scatterers and the electric sources are invariant along a co-ordinate, say the *z* axis. The problem for each illumination *v* ($v=1,.........V$) can be synthetically cast as:

$$E_{inc}^{v} = \mathcal{A}_p\left(J_p^{v}\right), \tag{4.a}$$

$$E^{v} - E_i^{v} = \mathcal{A}_i\left(\chi E^{v}\right), \tag{4.b}$$

$$E_s^{v} = \mathcal{A}_e\left(\chi E^{v}\right). \tag{4.c}$$

Equation (4.a) describes, by means of the integral operator $\mathcal{A}_p$ the relationship between the v-th primary sources J_p^v defined on a line Γ and the incident field E_i^v in the region under test Ω. In (4.b) $\chi(\mathbf{r}) = \varepsilon(\mathbf{r})/\varepsilon_b - 1$ is the contrast function between the permittivity of the isotropic (non magnetic) scatterer and that of the surrounding medium, E^v is the total field in Ω, and the integral operator $\mathcal{A}_i$ is the "internal" (as it acts from Ω to Ω) radiation operator. Finally (4.c) relates the induced currents in Ω represented by the χE^v product to the scattered field E_S^v as measured on Γ by means of the "external" radiation operator $\mathcal{A}_e$. Note primary sources and measurement probes are located, by the sake of simplicity, on the same curve Γ.

To achieve our alternative formulation it is convenient to take into account that (4.b) can be formally inverted as:

$$E^{v} = (I - \mathcal{A}_i\chi)^{-1} E_{inc}^{v}. \tag{5}$$

Then, let us observe that both the operators $\mathcal{A}_p$ and $\mathcal{A}_e$ are compact. Moreover, they share the same kernel but for the reversed roles played by integration and observation variables (as the former "goes" from Γ to Ω, and the latter "goes" from Ω to Γ). This means that one can define a triple $\{u_n, \sigma_n, v_n\}$ (n=1,..,∞) which is the Singular Value Decomposition (SVD) of $\mathcal{A}_e$ and that such a triple (but for reversed roles between the u_n and the v_n functions) is also the SVD of $\mathcal{A}_p$.

Now, exploiting the SVD properties, after some computations [4], one can express the field scattered in $\underline{r}_m$ due to an elementary primary source located in $\underline{r}_p$ as:

$$E_s\left(\underline{r}_m, \underline{r}_p\right) = \sum_{n=1}^{\infty}\sum_{h=1}^{\infty} S_h^n v_h\left(\underline{r}_m\right) v_n^*\left(\underline{r}_p\right) \tag{6}$$

where the elements S_h^n of the scattering matrix *S* have the formal expression

$$S_h^n = \sigma_n \sigma_h \left\langle \chi (I - \mathcal{A}_i\chi)^{-1} u_n, u_h \right\rangle. \tag{7}$$

For any $\underline{r}_m$ and for any $\underline{r}_p$, the scattered field can be evaluated from the knowledge of the matrix *S*. Moreover, since any primary source can be decomposed in a proper superposition of elementary sources, *S* encodes (for any given contrast) all possible scattering experiments.

Therefore, an alternative and equivalent way to formulate the inverse scattering problem at hand is that of retrieving χ from the knowledge of *S*. In addition to that, derivation of (6) and (7) also implies that a complete and convenient set of

scattering experiments can be obtained synthesizing the $v_n(\underline{r}_p)$ functions as primary sources and projecting the corresponding scattered fields over the $v_h(\underline{r}_m)$ functions set [4].
The above is also useful in achieving a regularization of the inverse problem. When Γ is placed some wavelenghts away from the scatterer, the singular values σ_n exhibit a step-like behaviour and, provided Ω is at least of the order of λ^2, the threshold N only depends on the electrical dimensions of the investigated area [4]. Moreover, the singular values decay exponentially fast after the knee N. Therefore, because of the unavoidable presence of measurement errors as well as to the necessarily finite precision in the realization of primary sources, contributions to the S matrix will be overwhelmed by errors for indices exceeding N. Accordingly, only a finite number of elements of S has to be taken into account to retrieve the contrast in a stable way. Due to reciprocity arguments, the number of independent parameters not overwhelmed by noise is $N^2/2$.Therefore, the number of parameters one can look for in the inverse problem is also bounded by $N^2/2$. These results are exploited in the next section to get a regularised formulation of the simple method.

3. THE TWO PROCESSING STEPS

In order to get an accurate and stable solution of the first step, id est, support estimation, let us translate the 'simple method' in a convenient finite dimensional framework. To this end, let us consider the v_n functions (n=1,..,P) as the set of independent primary sources. If the functions E_S^n are the corresponding (measured) scattered fields, the elements of the scattered matrix, as computed from data, are given by [4]:

$$S_h^n = \left\langle E_S^n(\varphi), v_h \right\rangle . \tag{8}$$

The equivalent "finite dimensional" formulation of (1) is to determine $A_n(y_0)$ such that:

$$\sum_{n=1}^{P} A_n(y_0) E_S^n(\varphi) = \exp[+ jk\rho \cos(\varphi - \alpha)] . \tag{9}$$

By projecting (9) onto v_h functions and exploiting (8) one gets:

$$\sum_{n=1}^{P} A_n(y_0) S_h^n = \langle \exp[+ jk\rho \cos(\varphi - \alpha)], v_h \rangle , \quad \forall h = 1,...P . \tag{10}$$

Translation into a finite dimensional frame is finally achieved by writing the counterpart of condition (3), that is:

$$\text{if} \quad y_0 \to \partial D \quad (y_0 \in D) \text{ then} \qquad \|\underline{A}(y_0)\| \to \infty , \tag{11}$$

As in its original infinite dimensional version (1-3) we come to a linear problem wherein the (herein finite dimensional) operator to be inverted contains the (error affected) data (see eq. 9). The two formulations are equivalent when P tends to infinity. Saying it in other words, the actual difference comes from the fact that we are considering a truncation of the scattering matrix rather than the whole scattering matrix (as implicitly in [3]). It is in fact well known that the "simple method" of [3] is subject to severe ill conditioning, so that some kind of regularization has to be used.
Hence the last step is the choice of P. Exploiting the results in Section II a natural and convenient choice is P=N; in fact, by virtue of the step-like behaviour of the singular values, small errors on the scattered fields would give rise to errors on the coefficients of the scattering matrix which have a negligible impact on those coefficients corresponding to $n \le N$, $h \le N$, while may have a relatively large impact on the remaining part of the scattering matrix (see eq. 7).
With such a choice we expect that our finite dimensional formulation achieves an increased robustness against ill-conditioning problems as compared with [3], which is in fact the case (see Sect. 4).

The information achieved with the approach described in the previous Section has now to be properly exploited in the framework of the "full" inversion of data, which is the secon step of the proposed procedure. This step is performed by looking for the global minimum of the non-linear functional

$$\sum_{v=1}^{V} \left\| E_i^v + \mathcal{A}_i\left(\hat{\chi}\hat{E}^v\right) - \hat{E}^v \right\|^2 \Big/ \left\|\hat{E}^v\right\|^2 + \left\| \mathcal{A}_e\left(\hat{\chi}\hat{E}^v\right) - y_S^v \right\|^2 \Big/ \left\| y_S^v \right\|^2 , \tag{12}$$

wherein y_S^n are the noise affected data, $\hat{\chi}$ and $\hat{E}^v$ are two finite dimensional representations of the unknowns. In particular a representation in terms of a Truncated Discrete Fourier Series is adopted for E^v, whereas χ is expanded in terms of wavelets. Such a last choice is particularly attractive in the present context. In fact, once the supports of the scatterers have been estimated, the information gained in the first step can be inserted in the inversion algorithm by simply selecting the wavelet coefficients corresponding to the topical sub-regions, while fixing the other ones to zero.
This procedure allows a considerable reduction of the number of unknown parameters. As noted in the Introduction and thoroughly discussed in [1], this has a beneficial effect on both ill conditioning and false solutions problems.

4. A NUMERICAL EXAMPLE AND CONCLUSIONS

To show effectiveness of the presented two-steps approach, let us present an example (which is representative of a large number of numerical simulations). In Fig.1.(a) the case is shown of two dielectric cylinders of side 0.5 m and permittivity 3.4 enclosed in a square region of side 1.7 m divided into 32x32 square cells. At 300 MHz, 16 incident plane waves and 16 receivers placed on a circle enclosing the scattering region are considered. Synthetic data are corrupted with a –30 dB Gaussian noise.
To retrieve the support of the scatterers we considered the scattered fields due to a set of primary sources given by the first nine (due to the value of N) v_n functions. In Fig. 1.(b) a contour plot of $\|A(y_0)\|$ in dB is given. The contours of the scatterers have been also been depicted. As the darker regions correspond to lower values of $\|A(y_0)\|$ this allows to identify the regions where the scatterers are located. Such a result is achieved in a few seconds on a common PC.
In the second step this information is exploited in the "full" inversion of data by considering only the wavelet coefficients corresponding to the dotted region highlighted in Fig.1.(b). By so doing we consider only 8 Daubechies-12 wavelets coefficients, achieving the reconstruction shown in Fig.1.(c). As no edge-preserving technique is used, the result is quite satisfactory. In particular, the reconstruction error is as low as 17%, whereas, when no support information is available, *no reconstruction is achieved at all*. It is also worth to note that, as expected, support estimation becomes much worse when a number of primary sources larger than N is used.

Given the success of the underlying idea, our activities are now devoted to the optimisation of each step, and to individuate further possible splitting of the overall processing. A first chance is given from the fact that estimation of the (possibly) non-convex support may take decisive advantage by a preliminary estimation of the convex hull of the scatterers (as solved in [2]). A second possibility comes from the fact that quantitative inversion is much more reliable and accurate when using an estimate of the mean value of the contrast as the starting guess [5]. Finally, post-processing techniques as the binary regularisation of [6] can furtherly refine accuracy of the solution in case a priori information are available on the contrast function one is looking for.

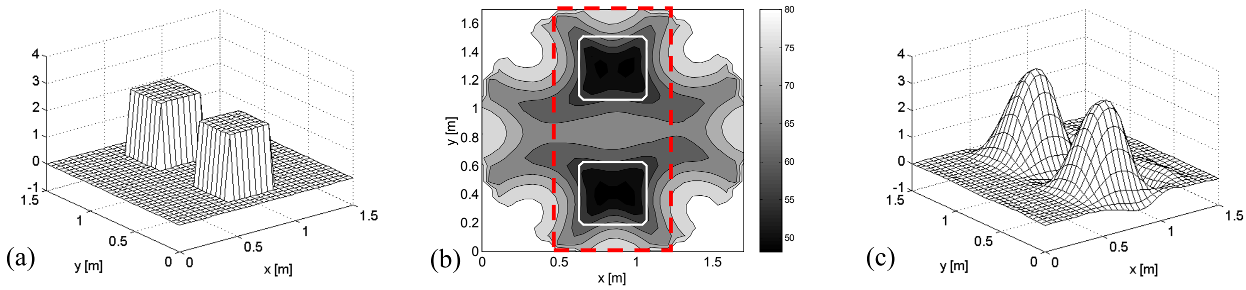

Fig.1: (a)Reference profile; (b) Plot of $\|A(y_0)\|$ in dB; (c) Reconstructed profile.

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