DIFFRACTION BY AN ANISOTROPIC IMPEDANCE HALF PLANE

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Abstract. For a plane wave incident at an oblique (skew) angle on an anisotropic impedance half plane, the second order difference equation for the angular spectra is converted to a pair of inhomogeneous non-singular integral equations and then solved. The exact expression for the scattered field is determined and compared with a previous approximate solution.

1 Introduction

The problem considered is that of a plane electromagnetic wave incident at an oblique (skew) angle on a half plane subject to first order anisotropic boundary conditions on the two faces. In contrast to the special cases of normal incidence or isotropic impedances, the application of Maliuzhinets' method [1] now leads to a second order difference equation for linear combinations of the angular spectra. This can be reduced to a pair of first order equations, but since these involve branched functions, the standard method of solution is no longer applicable. Somewhat similar first order equations have been considered by Senior and Legault [2] who showed how to solve them and construct, by combination, two linearly-independent solutions of the second order equation that are branch-free and have the desired properties. Unfortunately, the method involves elliptic integrals of the first and third kinds, and though in principle analytic, it is quite complicated and requires substantial computation to evaluate the functions that appear. Moreover, the particular equations that arise in the present problem have twice the number of branch points and have so far eluded solution, but based on approximations to the equations that have the effect of eliminating the branch points, Senior and Legault [3] obtained an approximate solution to the diffraction problem.

Recently a simpler and more direct method has been developed for the exact solution of second order difference equations of the type that occur here. The method [4] relies on a reduction in the period of the second order equation to make all three terms lie within the designated strip of analyticity of the solution. This permits the application of a modified Fourier transform in conjunction with the shift formula to produce an inhomogeneous non-singular integral equation which can be solved using only a few terms in a Taylor series for the unknown. The desired solutions are now trivial to obtain for any skew angle and impedances, and in the particular cases of normal incidence or skew incidence on an isotropic half plane, the solutions are in agreement with the known results. The exact expressions for the scattered field are derived. The backscattered far field amplitude is computed for specimen impedances and a variety of skew angles, and the results are compared with the approximate solution in [3].

2 Formulation

The 4π periodic second order difference equation satisfied by $t_{\mathrm{e}}(\alpha)$ and $t_{\mathrm{h}}(\alpha)$ is [5]

$$\bar{t}(\alpha + 6\pi) + \bar{a}(\alpha)\, \bar{t}(\alpha + 2\pi) + \bar{t}(\alpha - 2\pi) = 0 \tag{1}$$

with

$$\overline{a}(\alpha) = -2 - \frac{\left\{2\left(\frac{1}{\eta_1} - \eta_2\right)\frac{\sin\alpha}{\sin\beta}\right\}^2}{(\sin^2\alpha - \sin^2\theta_1)(\sin^2\alpha - \sin^2\theta_2)}\,. \tag{2}$$

Equation (1) has the same form as that considered by Senior et al. [4], and in the manner shown there, (1) can be reduced to the 2π periodic difference equation

$$\overline{t}(\alpha + 2\pi) + c(\alpha)\,\overline{t}(\alpha) - \overline{t}(\alpha - 2\pi) = 0 \tag{3}$$

where $c(\alpha)$ is a 4π periodic function such that

$$c(\alpha)\; c(\alpha + 2\pi) = \frac{\left\{2\left(\frac{1}{\eta_1} - \eta_2\right)\frac{\sin\alpha}{\sin\beta}\right\}^2}{(\sin^2\alpha - \sin^2\theta_1)(\sin^2\alpha - \sin^2\theta_2)}\,. \tag{4}$$

The particular form of (3) required for $t_{\mathrm{e}}(\alpha)$ is therefore

$$\overline{t}(\alpha + 2\pi) + c_{\mathrm{e}}(\alpha)\,\overline{t}(\alpha) - \overline{t}(\alpha - 2\pi) = 0 \tag{5}$$

where, after some simplification,

$$c_{\mathrm{e}}(\alpha) = -\frac{\frac{1}{2}\left(\frac{1}{\eta_1} - \eta_2\right)\frac{\sin\alpha}{\sin\beta}}{\left(\cos\frac{\alpha}{2} + \sin\frac{\theta_1}{2}\right)\left(\cos\frac{\alpha}{2} + \cos\frac{\theta_1}{2}\right)\left(\cos\frac{\alpha}{2} - \sin\frac{\theta_2}{2}\right)\left(\cos\frac{\alpha}{2} - \cos\frac{\theta_2}{2}\right)} \tag{6}$$

and it is evident that this satisfies (4). Similarly, for $t_{\mathrm{h}}(\alpha)$,

$$\overline{t}(\alpha + 2\pi) + c_{\mathrm{h}}(\alpha)\,\overline{t}(\alpha) - \overline{t}(\alpha - 2\pi) = 0 \tag{7}$$

with

$$c_{\mathrm{h}}(\alpha) = \frac{\frac{1}{2}\left(\frac{1}{\eta_1} - \eta_2\right)\frac{\sin\alpha}{\sin\beta}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\theta_1}{2}\right)\left(\cos\frac{\alpha}{2} - \cos\frac{\theta_1}{2}\right)\left(\cos\frac{\alpha}{2} + \sin\frac{\theta_2}{2}\right)\left(\cos\frac{\alpha}{2} + \cos\frac{\theta_2}{2}\right)}\,. \tag{8}$$

3 Solution of the Difference Equations

As shown in [4], (5) can be written as

$$\overline{t}(\alpha) - \frac{1}{2}\left\{\overline{t}(j\infty) + \overline{t}(-j\infty)\right\} = \frac{j}{8\pi}\int_{-j\infty}^{j\infty} c_{\mathrm{e}}(\alpha')\,\overline{t}(\alpha')\;\tan\frac{1}{4}|\alpha - \alpha'|\,d\alpha' \tag{9}$$

with a similar result for (7), and the second term on the left hand side is necessary to take account of the finite values of $\overline{t}(\alpha)$ at $\alpha = \pm j\infty$. Since $c_{\mathrm{e}}(\alpha)$ is an odd function and $\overline{t}_{\mathrm{e}}(\alpha)$ is even, a more compact version of (9) is

$$\overline{t}(\alpha) - \overline{t}(j\infty) = -\frac{j}{4\pi}\int_0^{j\infty} c_{\mathrm{e}}(\alpha')\,\overline{t}(\alpha')\,\frac{\sin\frac{\alpha'}{2}}{\cos\frac{\alpha}{2} + \cos\frac{\alpha'}{2}}\,d\alpha', \tag{10}$$

which is an inhomogeneous integral equation for $\bar{t}(\alpha)$ for positive imaginary α. Apart from a normalization factor, $\bar{t}(j\infty)$ is specified by the edge behavior and plays the role of an equivalent excitation.

It is convenient to map the path of integration in (10) on to the real x axis by writing $x = -j\tan(\alpha'/4)$, in which case

$$d\alpha' = \frac{4j}{1-x^2}\,dx$$

with $x=1$ corresponding to $\alpha' = j\infty$ and $x=0$ to $\alpha'=0$. Hence, if $\bar{t}(\alpha) = f(-j\tan\frac{\alpha}{4}) = f(y)$,

$$f(y) - f(1) = \frac{2}{\pi}\left(\frac{1}{\eta_1} - \eta_2\right)\,\frac{1}{\sin\beta}\int_0^1 x^2(1-x^4)\,g(x)\,f(x)\,\frac{1-y^2}{1-x^2y^2}\,dx, \tag{11}$$

which is an inhomogeneous integral equation for $f(y)$ valid for $0 \le y \le 1$. Here

$$g(x) = \left[\left\{x^2\left(1-\sin\frac{\theta_1}{2}\right)+1+\sin\frac{\theta_1}{2}\right\}\left\{x^2\left(1-\cos\frac{\theta_1}{2}\right)+1+\cos\frac{\theta_1}{2}\right\}\right.$$
$$\left.\cdot\left\{x^2\left(1+\sin\frac{\theta_2}{2}\right)+1-\sin\frac{\theta_2}{2}\right\}\left\{x^2\left(1+\cos\frac{\theta_2}{2}\right)+1-\cos\frac{\theta_2}{2}\right\}\right]^{-1}. \tag{12}$$

The function $f(y)$ is a real slowly varying function of y which can be represented by the Taylor series

$$f(y) = \sum_{n=0}^{\infty} a_n y^{2n} \tag{13}$$

with $a_0 = 1$ in accordance with the normalization. This suggests a method for the solution of (11) for all β. On inserting (13) into (11) we obtain

$$\sum_{n=1}^{\infty} a_n\left\{h_n(y) - y^{2n} + 1\right\} = -h_0(y) \tag{14}$$

where

$$h_n(y) = \frac{2}{\pi}\left(\frac{1}{\eta_1} - \eta_2\right)\,\frac{1}{\sin\beta}\int_0^1 x^{2n+2}(1-x^4)\,g(x)\,\frac{1-y^2}{1-x^2y^2}\,dx. \tag{15}$$

Only a handful of terms in the Taylor series is sufficient to produce good accuracy.

In Figure 1 the backscattered ($\phi = \phi_0$) far field amplitude is plotted as a function of ϕ_0, $0 \le \phi_0 \le \pi$, for $\eta_1 = 2$, $\eta_2 = 4$, and $\beta = \pi/2$ (normal incidence), $\pi/3$, $\pi/6$, and $\pi/12$. The approximate solution obtained by Senior and Legault [1998] is also shown and the agreement is remarkable.

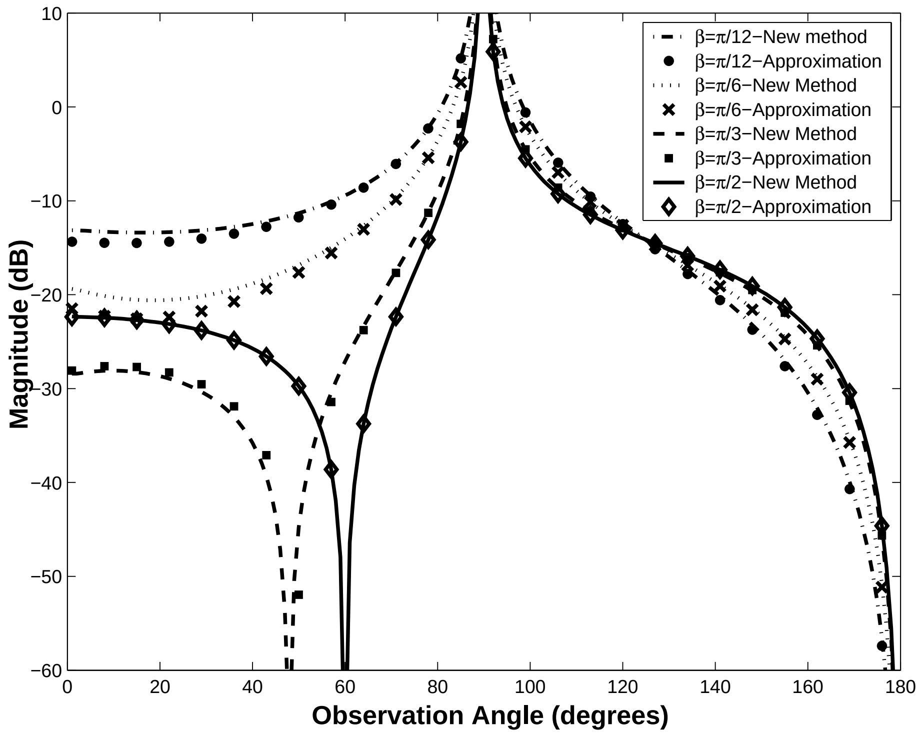

Figure 1: Backscattered far field amplitude for $\eta_1 = 2$ and $\eta_2 = 4$, compared with the approximate solution.

References

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