NONLINEAR DYNAMICS OF SHORT PULSES IN OPTICAL FIBRES WITH STRONG TRANSVERSE AND WEAK LONGITUDINAL INHOMOGENEITIES

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ABSTRACT

Short optical pulse propagation in graded-index light guides is modelled with a three-dimensional nonlinear wave equation taking into account a Kerr-like nonlinearity, different inhomogeneity scales in transverse and longitudinal directions, and either curvature and twist of the fibre axis. The linear and nonlinear parts of the propagation process are regularly separated in the course of a consistent asymptotic procedure. The set of propagating modes is described with a singular Sturm-Liouville problem, and the pulse envelope is shown to obey a nonlinear Schrödinger equation with coefficients depending on the longitudinal coordinate and expressed through the characteristics of the transverse inhomogeneity.

INTRODUCTION

Propagation of a short optical pulse in a light guide can be treated as a striking example of a nonlinear electromagnetic wave process localized within a bounded volume [1], the localization with respect to different directions resulting from different factors. The medium of propagation is characterized with different inhomogeneity scales in the longitudinal direction and in the cross-section of the fibre. So far as the amplitude of the electromagnetic field in the pulse turns out significantly smaller than the threshold of self-focusing [2] then the propagation process should be regarded as a weak nonlinear one. This allows to introduce a small parameter, in view of further asymptotic description of the process, and to set a hierarchy of subprocesses and scales.

As is shown in the present paper, the weak nonlinearity of the process does not affect the transverse concentration of the wave field, which is provided at the expense of the strong transverse inhomogeneity of the fibre. In contrast to that, the concentration in the longitudinal direction is wholly due to the nonlinearity [3], and it leads to formation of the envelope soliton [4].

The propagation of short pulses is modelled with a nonlinear wave equation [5] written down in the vicinity of a spatial curve, a fibre axis, characterized with given curvature and twist

$$\Delta f-(\beta^2(\rho,\varphi,s)+\frac{1}{2}\alpha(\rho,\varphi,s)<f^2>)\frac{\partial^2 f}{\partial t^2}=0 \tag{1}$$

f is a component of the electromagnetic field, $<f^2>$ means the square of the field averaged over the period of oscillations, ρ, φ, s are dimensionless coordinates, and t is a dimensionless time. The field, f, is supposed to be on the order of $\delta<<1$, the longitudinal coordinate, s, is taken proportional to δ^2 manifesting the weak longitudinal inhomogeneity of the fibre, and ρ,φ are the conventional polar coordinates in the cross-section. Under such a choice of the longitudinal inhomogeneity scale, the curvature and twist of the axis are natural to be set as quantities on the order of δ^2 as well. So far as the wave field under investigation is supposed to be localized the solution of (1) must vanish for $\rho\to\infty$.

ANSATZ

Let us introduce the phase of the pulse envelope with the formula

$$\vartheta=\frac{Q(s)}{\delta}-\delta t$$

and seek for the solution to (1) as

$$f = \delta F(\rho,\vartheta,s,\varphi)e^{i(\frac{R(s)}{\delta^2}-t)} + \delta\bar{F}(\rho,\vartheta,s,\varphi)e^{-i(\frac{R(s)}{\delta^2}-t)} \tag{2}$$

The main feature of the *ansatz* used here is the difference between the phases of the envelope and carrier. The complex amplitude, F , is expanded into series by powers of the small parameter δ

$$F(\rho,\vartheta,s,\varphi) = \sum_{j=0}^{\infty} \delta^j F_j(\rho,\vartheta,s,\varphi) \tag{3}$$

F_o and F_1 are possible to be independent on the azimuth angle, φ, but even in the case of independence of β and α on φ , a dependence on the azimuth angle does arise starting with the term F_2 , which appears to be a consequence of the bending of the fibre axis.

TRANSVERSE LOCALIZATION OF THE FIELD

Substituting expansions (2), (3) into (1) and equating zero the terms on the same order of δ result in a series of boundary-value problems for second-order differential equations which, being combined with conditions of their solvability, permit to determine all the elements of the *ansatz*.

At the major order one obtains the equation

$$\frac{\partial^2 F_0}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F_0}{\partial\rho} + \left(\beta^2(\rho,s,\varphi) - r^2(s)\right)F_0 + \frac{1}{\rho^2}\frac{\partial^2 F_0}{\partial\varphi^2} = 0 \tag{4}$$

If the linear part of the refractive index, β, is azimuth-symmetric then the variables in (4) can be separated. The function F must be periodic with respect to φ with the period of 2π, therefore $F_0 \propto e^{im\varphi}, \frac{\partial^2 F_0}{\partial\varphi^2} = -m^2 F_0$, m is an integer number. An azimuth symmetric solution can be obtained formally for $m=0$. Regarded as a function of ρ and s, F_0 obeys the equation

$$\frac{\partial^2 F_0}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F_0}{\partial\rho} + \left(\beta^2(\rho,s) - r^2(s) - \frac{m^2}{\rho^2}\right)F_0 = 0 \tag{5}$$

and the boundary conditions

$$\frac{\partial F_0}{\partial\rho}\Big|_{\rho=0} = 0, \quad F_0 \underset{\rho\to\infty}{\to} 0 \tag{6}$$

The problem in (5), (6) appears to be a singular Sturm-Liouville problem which has an additional difficulty conditioned by the fact that not only the interval of definition $\rho \in [0,\infty)$ is semi-infinite but also the coefficients of the equation have non-integrable singularity at $\rho=0$. The approach [6] of investigation of the Sturm-Liouville problem with regular coefficients over the infinite interval $(-\infty,\infty)$ can be altered to the problem in (5), (6). It means that, for a sufficiently wide and practically important class of functions $\beta(\rho,s)$, the problem in (5), (6) has a series of eigenvalues $r_j^2(s)$ and corresponding to them eigenfunctions F_{0j}. It is the set from that the principal mode (2) propagating in the light guide is chosen.

Equation (5) implies that

$$F_0(\rho, \vartheta, s, \varphi) = V(\rho, s)U(\vartheta, s)\, e^{im\varphi} \tag{7}$$

where $V(\rho,s)$ is a chosen eigenfunction of the problem in (5), (6), normalized by the condition

$$\int_0^\infty \rho V^2(\rho, s) d\rho = 1 \tag{8}$$

and the function U characterizes the envelope of the principal mode in the major approximation. Later on this will be simply referred to as the pulse envelope.

The problem in (5), (6) has explicit solutions in the case of a quadratic dependence of β^2 on the radial coordinate ρ

$$\beta^2(\rho, s) = \beta_0^2(s) - \frac{\beta_2^2(s)}{4}\rho^2$$

By means of the direct substitution one can be convinced that the j-th eigenvalue turns out

$$\left(r_j^{(m)}(s)\right)^2 = \beta_0^2(s) - (2j + m + 1)\beta_2(s)$$

and the corresponding eigenfunction $V_j^{(m)}$ normalized by means of (8) is represented by

$$V_j^{(m)}(\rho, s) = \left(\frac{\beta_2(s)}{2}\right)^{\frac{2m+1}{4}} \rho^m\, e^{-\frac{\beta_2(s)\rho^2}{4}} L_j^{(m)}\left(\frac{\beta_2(s)\rho^2}{2}\right)$$

where $L_j^{(m)}$ – the generalized Laguerre polynomials.

PULSE ENVELOPE SOLITON

The compatibility condition for the second-order differential equation for F_2 completed with the proper boundary conditions consists in the orthogonalaty with the weight ρ of its right-hand side to the function V, the solution of the homogeneous problem in (5), (6). Taking into account the presentations for F_0 and F_1 one can obtain an equation for the pulse envelope

$$2ir(s)\frac{\partial U}{\partial s} + g(s)\frac{\partial^2 U}{\partial \vartheta^2} + ir'(s)U + h(s)\,|\,U\,|^2\, U = 0 \tag{9}$$

the nonlinear Schrödinger equation with coefficients depending on the longitudinal coordinate. These coefficients are expressed by the formulae

$$g(s) = 4\int_0^\infty \rho\left(\beta^2(\rho,s) - q(s)r(s)\right)V(\rho,s)W(\rho,s)d\rho - \int_0^\infty \rho\left(\beta^2(\rho,s) - q^2(s)\right)V^2(\rho,s)d\rho$$

$$h(s) = \int_0^\infty \rho\alpha(\rho,s)V^4(\rho,s)d\rho$$

where $W(\rho,\ s)$ is the solution of the problem for F_1 and describes the transverse distribution of the field in the first-order correcting term to the complex amplitude.

Dependence of the coefficients in (9) on s infers from the longitudinal inhomogeneity and reflects its influence on the pulse envelope evolution in the light guide. In the case of the longitudinal inhomogeneity of a special kind, namely

$$2g(s)r(s) = p^2 h(s) \quad , \quad p = \text{const}$$

a soliton solution to (9) can be written down in elementary functions

$$U(\vartheta, s) = \frac{p}{\sqrt{r(s)}} \frac{e^{i\vartheta}}{\text{ch}\left[\vartheta - \int_0^s \frac{g(s')}{r(s')} ds'\right]} \tag{10}$$

which, in combination with (2), obviously illustrates the three-scale character of the propagation of the short pulse : the high-frequency carrier is modulated with the envelope which evolves in a two-scale manner, the medium-rate evolution of the envelope phase is accompanied with a slow-rate variation of the amplitude.

CONCLUSION

The main content of the present work consists in modelling the process of short optical pulse propagation in the graded-index light guide by means of a nonlinear wave equation, and in applying a consecutive asymptotic procedure to its solution. The fiber refractive index is assumed to vary slowly in the longitudinal direction and furthermore a slight spatial bending of the fiber axis is taken into account as well. The propagation process is regarded as weak nonlinear, which permits, within the framework of the asymptotic procedure applied, to separate naturally the problems on wave field distribution in the light guide's cross-section and on pulse propagation as itself. The former problem is a linear one, concentration of the wave field in the vicinity of the light guide axis is provided by the very dependence of the refractive index of the fiber on the radial coordinate. The set of all the modes which can propagate in the graded-index light guide with a given profile of the refractive index is defined with the singular Sturm-Liouville problem in (5), (6). In the case of the quadratic dependence of the linear part of the second power of the refractive index on the radial coordinate, the transverse distribution of the field is expressed in the explicit form, even those modes included for which the field distribution in the cross-section depends on the azimuth angle.

The pulse propagation as itself appears to be a nonlinear process that is shown to have a three-scale character. The fast subprocess – high-frequency carrier propagation – is modulated with the envelope which meets the nonlinear Schrödinger equation with coefficients depending on the longitudinal coordinate. The envelope evolution is two-scale and formed with the evolution of the envelope phase and the slow variation of the pulse amplitude, it can be demonstrated for a certain narrow class of the longitudinal inhomogeneity for which a soliton solution to equation (9) is expressed explicitly (10). It's worth while noting that the formulae for the coefficients include the characteristics of the transverse inhomogeneity and the transverse distribution of the wave field of the propagating mode, then they are different for different modes. It means that the technique proposed permits to describe the pulse dynamics, with its mode structure being fully taken into account.

ACKNOWLEDGEMENT

This work was supported by the Russian Foundation for Basic Research under grant № 02-02-16046.

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