A Closed Expression for the Maximum Unambiguous Angular Segment of Nonuniform Linear Arrays

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ABSTRACT

A crucial point in the design of antenna arrays is to avoid the occurrence of ambiguities also known as grating lobes. The signals from an ambiguous and the main beam direction result in identical array output and can not be distinguished. For arrays of uniform, i.e. equi-distant single elements the occurrence and position of ambiguities can easily be formulated in a closed expression. The unambiguous angular segment is the angular segment between the direction of the main beam and the first ambiguity. This paper derives an expression for the unambiguous angular segment for the more general case of linear arrays with arbitrary single elements' position.

I. INTRODUCTION

Antenna arrays used in radar or beamforming applications impose fundamental limitations concerning the resolution and unambiguous angular segment of the systems. Two contradicting requirements have to be met, the resolution (given by the 3dB beamwidth in conventional beamforming processing) is determined by the extend of the array. The unambiguous angular segment Ψ^{unamb} however, is determined by the appearance of grating lobes, resulting in restrictions on the maximum distance between any two neighbouring antenna elements in the array. To fulfil both requirements, i.e., high resolution and a large Ψ^{unamb}, a long array with a large number of elements is required, which results in high cost and complexity.

An attractive method to increase the unambiguous angular segment by suppressing the grating lobes are nonuniform arrays. The resolution is still determined by the extend of the array, however, the number of elements can be reduced without causing any ambiguities. This is especially useful in applications with tight space requirement for the antenna.

The paper derives a closed mathematical expression for the unambiguous angular segment (UAS) as a function of the elements' positions. Previous research activities in this field have been focused on the statistical placement in one or two dimensions of a large number of single elements so as to yield an array pattern within given boundary limits [1]. However, these methods are not deterministic and yield good results only for large arrays (tens or hundreds of single elements). The solution proposed here does not require any lower bound on the number of elements.

II. ANTENNA CONFIGURATION

The array under consideration is composed of N single antenna elements located at $x_1, x_2, \dots x_N$ along the $x-$axis as shown in Fig.1. Consider a transmitter or equivalently a scatterer located at r_s, ψ_s. The signal u_n received from element n due to a narrow-band plane wave impinging on the array at an angle ψ_s is given by

$$u_n = a(r_s) \cdot C_n(\psi_s) \cdot e^{j(\omega t - k s_n)} \quad , \quad n = 1, \dots, N \tag{1}$$

The constant $k = 2\pi/\lambda$ is the wavenumber. The normalised receive pattern of antenna element n is given by $C_n(\psi_s)$. The —angle independent term— $a(r_s)$ accounts for the attenuation of the signals and is assumed identical for all receivers. The distance term s_n is found from elementary geometry to be

$$s_n = \sqrt{r_s^2 - 2r_s x_n \sin\psi_s + x_n^2} \tag{2}$$

Expressing the square-root as a Taylor series and considering only the linear term yields

$$s_n \approx r_s - x_n \sin\psi_s \tag{3}$$

it is well known that the above approximation is valid for most practical cases, detailed investigation for radar applications can be found in [2].

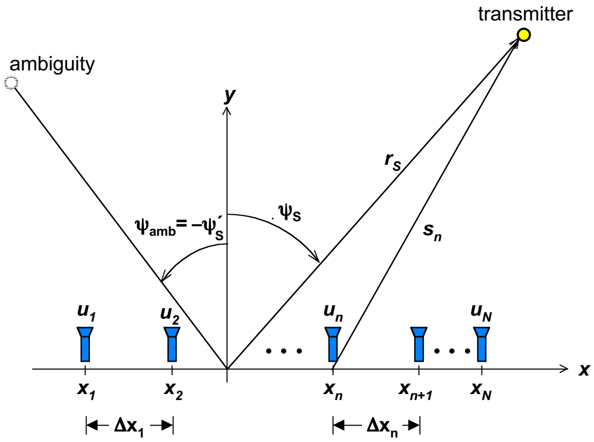


Fig. 1. Geometry for linear array of receivers

III. AMBIGUITY CONDITION

An ambiguity occurs if-and-only-if two plane waves impinging on the array from different angles result in identical received signals for *all* array elements. For a linear array of isotropic elements, i.e. $C_n \equiv 1$, $\forall\, n = 1 \ldots N$ a front-back ambiguity exists, i.e. $\psi_{\text{amb}} = 180^o - \psi_s \quad \forall\, \psi_s$ which is independent of the array configuration. To derive a general expression only ambiguities within $-\pi/2 \leq \psi_s \leq \pi/2$ will be considered.

Consider the received signals due to a plane wave induced by a transmitter located at (r_s, ψ_s)

$$u_n = a(r_s) \cdot C_n(\psi_s) e^{-jks_n} \quad , \quad n = 1, \ldots, N \tag{4}$$

and a second transmitter at (r_s', ψ_s') with

$$u_n' = a(r_s') \cdot C_n(\psi_s') e^{-jks_n'} \quad , \quad n = 1, \ldots, N \tag{5}$$

in the above the exponential frequency term as it appears in (1) is suppressed. The aim is to determine the smallest angle difference between the two transmitters which result in an ambiguity. The conditions for an ambiguity can be stated as follows

1. Phase Condition The absolute phase of the received signals is not relevant. It is required that adjacent antennas have identical *phase differences* modulo an integer of 2π

$$\angle u_n - \angle u_{n+1} \stackrel{!}{=} \angle u_n' - \angle u_{n+1}' + m_n \cdot 2\pi \quad , \quad m_n \in \mathbb{Z}; \quad n = 1, \ldots, N-1 \tag{6}$$

2. Amplitude Condition The amplitude of the received signals should be identical for both transmitters excluding a constant factor which is due to possible difference in distance and independent of the receive element

$$\frac{|u_n|}{|u_{n+1}|} \stackrel{!}{=} \frac{|u_n'|}{|u_{n+1}'|} \quad , \quad n = 1, \ldots, N-1 \tag{7}$$

for $|u_n|, |u_n'| \neq 0 \quad \forall \quad n = 2, \ldots, N$.

Combining conditions 1. and 2. is equivalent to the requirement of equal ratios of the received complex signals

$$\frac{u_n}{u_{n+1}} \stackrel{!}{=} \frac{u_n'}{u_{n+1}'} \quad , \quad n = 1, \ldots, N-1 \tag{8}$$

which when inserting (1) together with the approximation (3) and after simplification yields

$$\frac{C_n(\psi_s)}{C_{n+1}(\psi_s)} e^{jk(x_n - x_{n+1})\sin\psi_s} \stackrel{!}{=} \frac{C_n(\psi_s')}{C_{n+1}(\psi_s')} e^{jk(x_n - x_{n+1})\sin\psi_s'} \quad , \quad n = 1, \ldots, N-1 \tag{9}$$

The above gives a general condition for the occurrence of an ambiguity, i.e. two angles $\psi_s \neq \psi_s'$ are ambiguous if for a given linear array configuration (9) is satisfied. The phase condition is related to the exponential terms and thus to the angle of incidence. The amplitude condition is related to the —angle dependent— wheightening imposed by the radiation pattern of the individual elements. It should be noted that condition (9) is *not* a function of the distance r_s.

IV. UNAMBIGUOUS ANGULAR SEGMENT (UAS)

An array consisting of non-identical single elements will have an unambiguous angular segment larger than the corresponding array of identical elements. According to (9) this is because in addition to the phase condition, given by the exponential terms, the amplitude condition has to be satisfied. A closed expression for the UAS can only be obtained assuming identical radiation patterns, i.e. $C_1(\psi) = C_2(\psi) = \ldots = C_N(\psi)$. Then (9) simplifies to

$$\begin{aligned} & e^{-jk\Delta x_n \sin\psi_s} = e^{-jk\Delta x_n \sin\psi'_s} && , \quad n = 1,\ldots,N-1 \\ \Rightarrow \quad & \frac{2\pi}{\lambda}\left(\sin\psi_s - \sin\psi'_s\right)\Delta x_n = m_n \cdot 2\pi && , \quad m_n \in \mathbb{Z}; \quad n = 1,\ldots,N-1 \end{aligned} \tag{10}$$

where the separation of the antenna elements is written as

$$\Delta x_n = x_{n+1} - x_n \quad , \quad n = 1,\ldots,N-1 \tag{11}$$

The array is assumed to be scanned to the angle ψ_s and the smallest value for $|\psi'|$ gives the angle of the first ambiguity. For $0 \le \psi_s \le \pi/2$ the first ambiguity will always occur for $\psi' \le 0$ since this results in the largest value for the difference $|\sin\psi_s - \sin\psi'_s|$ in (10). Denoting $\psi_{amb} = -\psi'$ and substituting into (10) yields

$$\frac{\Delta x_n}{\lambda}(\sin\psi_s + \sin\psi_{amb}) = m_n \quad , \quad m_n \in \mathbb{Z}; \quad n = 1,\ldots,N-1 \tag{12}$$

The condition (12) remains the same for $-\pi/2 \le \psi_s \le 0$ and $0 \le \psi'$. For a uniform linear array, i.e. $\Delta x_n = \Delta x$, $\forall n$ the above simplifies to the well known condition for the occurrence of grating lobes in phased arrays [3].

Since the $\sin$-function is monotone within the considered angular segment, the smallest ψ_{amb}, is determined by the smallest $\sin\psi_{amb}$ satisfying (12). An ambiguity is found by determining the integers $m_1,\ldots,m_{N-1}$ which together with the separations $\Delta x_1, \ldots \Delta x_{N-1}$ result in identical ψ_{amb}. The UAS Ψ^{unamb} is given through the smallest ψ_{amb} by

$$\Psi^{unamb} = |\psi_s + \min\{\psi_{amb}\}| \tag{13}$$

the $\min\{\psi_{amb}\}$ is determined by the *smallest* integers m_n which together with Δx_n fulfil (12) for all n.

The normalised distances between the elements are written as rational relatively prime integer numbers

$$\frac{\Delta x_n}{\lambda} = \frac{a_n}{b_n} \quad , \quad a_n, b_n \in \mathbb{Z}; \quad b_n \neq 0; \quad \text{relatively prime;} \quad \mathrm{n} = 1,\ldots,\mathrm{N}-1 \tag{14}$$

In addition

$$\sin\psi_s + \sin\psi_{amb} = \frac{c}{d} \quad , \quad c, d \in \mathbb{Z}; \quad d \neq 0; \quad \text{relatively prime} \tag{15}$$

Presuming rational integers does not impose any practical restriction on the quantities to the left of the equal sign in (14) or (15). Substituting into (12) and rearranging terms

$$\frac{c}{b_n} \cdot \frac{a_n}{d} = m_n \quad , \quad m_n \in \mathbb{Z}; \quad n = 1,\ldots,N-1 \tag{16}$$

Since a_n/b_n and c/d are both relatively prime, each of the two fractions in the above expression must result in an integer to satisfy (16). The approach is to determine the smallest values for the two fraction.

First fraction:

$$\frac{c}{b_n} \overset{!}{\in} \mathbb{Z}; \quad n = 1,\ldots,N-1 \tag{17}$$

It is obvious that c has to be a common multiple of $b_1,\ldots,b_{N-1}$. The smallest value for the fraction is given by the smallest possible c. Thus c is the least common multiple (LCM) of $b_1,\ldots,b_{N-1}$

Second fraction:

$$\frac{a_n}{d} \overset{!}{\in} \mathbb{Z}; \quad n = 1,\ldots,N-1 \tag{18}$$

Here d should be a common divisor of $a_1, \ldots, a_{N-1}$. The smallest value for the fraction is given by greatest value for d. Thus d is the greatest common divisor (GCD) of $a_1, \ldots, a_{N-1}$

Substituting into (15) results in the expression for the ambiguous angle

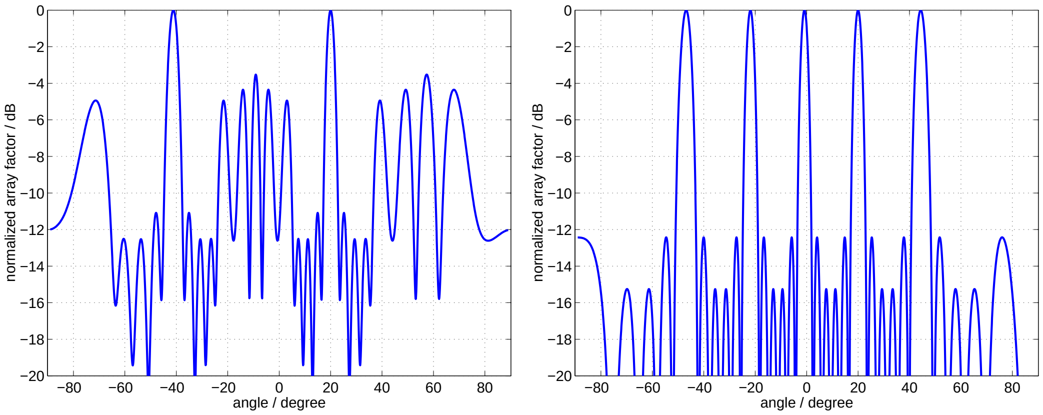
$$\psi_{amb} = \arcsin\left(\frac{LCM\,\{b_1, \ldots, b_{N-1}\}}{GCD\,\{a_1, \ldots, a_{N-1}\}} - \sin\psi_s\right) \tag{19}$$

As would be expected the above expression is a function of *all* the separations between the elements and not just the smallest separation between two elements in the array.

V. VERIFICATION AND RESULTS

Expression (19) and (13) are used to calculate the UAS for a given non-uniform array. It should be noted that it is possible to find values for Δx_n resulting in $\sin\psi_{amb} > 1$ which corresponds to an ambiguous free configuration. However it turns out that such configurations most often result in high side-lobe levels and are thus of no practical use.

Consider a linear non-uniform array of $N = 6$ with $x_n = 0\lambda, 2\lambda, 4\lambda, 7\lambda, 10\lambda, 14\lambda$ scanned to an angle $\psi_s = 20^o$. The ambiguity angle and the UAS are found to be $\psi_{amb} = -\psi'_s = 41.1^o$ and $\Psi^{unamb} = 61.1^o$ respectively (note that ψ_s and ψ'_s are of opposite sign). The array factor of the configuration shown in Fig. 2(a) confirm these values. Note that the smallest separation $\min\{\Delta x_n\} = 2\lambda$ would result in an UAS of 29^o which is smaller than for the non-uniform array. For comparison Fig. 2(b) shows the array factor of a uniform array having the same number of elements ($N = 6$) and span $x_{max} - x_{min} = 14\lambda$. Both array factors have a main beamwidth of 3.4^o but the uniform array results in an UAS of 20.9^o which is considerably less than for the non-uniform array.



(a) positions $x_n = 0\lambda, 2\lambda, 4\lambda, 7\lambda, 10\lambda, 14\lambda$

(b) positions $x_n = 0\lambda, 2.8\lambda, 5.6\lambda, 8.4\lambda, 11.2\lambda, 14\lambda$

Fig. 2. Array factor for a 6 element non-uniform (a) and uniform (b) linear array

VI. CONCLUSION

The paper derives a closed expression for the minimum unambiguous angular segment. The solution is valid for linear arrays of arbitrary spaced non-identical single elements covering a common angular segment. In the special case of equi-distant and identical single elements the expression converges to the well-known form for phased arrays.

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