ON SOLUTION OF PARABOLIC MARKOV'S EQUATION FOR SPACE-FREQUENCY COHERENCE FUNCTION

N.N.Zernov[(1)1], **V.E.Gherm**[(2)1]

[(1)] ***University of St.Petersburg, St.Petersburg, Russia***

Institute of Radiophysics, Ulyanovskaya, 1, Petrodvorets 198504, St.Petersburg, Russia

zernov@paloma.spbu.ru

[(2)] ***As (1) above, but E-mail: gherm@paloma.spbu.ru***

ABSTRACT. Asymptotic technique to solve Markov's parabolic equation for the second order spaced position and frequency coherence function is discussed. Rather than employing separation of variables, the technique is based on the quasi-classic representation in terms of complex trajectories and is also valid in the case of a non-homogeneous background medium and does not demand the statistical homogeneity of fluctuations. It has no constraints relevant to the initial conditions in the form of an incident plane wave and produces in automatic fashion different known rigorous solutions, in particular, to the case of quadratic structure function.

INTRODUCTION

Despite of many years of exploitation of Markov's parabolic equation there still exist the problems in constructing the second order coherency in the case of spaced frequency. The problem is to produce the solution to the case of the structure function of the fluctuations different than the quadratic. The classic solution to this case given in [1], does not allow further extension to more general models of the structure function. In the recent years the major progress here has been achieved by Oz and Heyman [2], who constructed the second order time-space coherency by means of separation of variables. Their approach implies the expansion of the solution into the series of the transversal eigenfunctions of the problem and sometimes faces the difficulties, in particular, the series in the expansions fails to converge for some types of initial conditions (e.g. an incident plane wave) as the distance and difference frequency tend to zero. Additional constraints in the separation of variables technique may also arise when considering fluctuations with the realistic structure function tending to a constant as difference variable tends to the infinity. In this case the continuous spectrum may likely occur in the spectrum of the transversal operator of the problem that makes expansion of the solution in terms of the transversal eigenfunctions much more complicated. The technique presented here is free of the enumerated constraints.

MAIN EQUATIONS AND RELATIONSHIPS

Classic two-frequency second order Markov's parabolic equation [3] is considered for the homogeneous background medium, where possible statistical non-homogeneity of fluctuations in the longitudinal direction is allowed as follows

$$K\frac{\partial\Gamma_1}{\partial\varsigma}+\frac{i}{2}\left[\tilde{k}_d\nabla^2_{\mathbf{r}}+\frac{\tilde{k}_d}{4}\nabla^2_{\mathbf{R}}-2\tilde{k}_s\nabla_{\mathbf{R}}\nabla_{\mathbf{r}}\right]\Gamma_1+\frac{K^2}{4}\left[\tilde{A}(\varsigma,0)-\tilde{A}(\varsigma,\mathbf{r})\right]\Gamma_1=0\,. \tag{1}$$

Here the dimensionless variables are introduced according to: $z=\zeta\, l_\varepsilon$, $\mathbf{r}_-=\mathbf{r}\, l_\varepsilon$ and $\mathbf{r}_+=\mathbf{R}l_\varepsilon$, where $\mathbf{r}_-$ and $\mathbf{r}_+$ are the transversal difference and central variables respectively; l_ε is the effective spatial scale of fluctuations. Parameter $K=k_1k_2l_\varepsilon^2$ is the dimensionless parameter, which is assumed to be the large parameter of the problem; $\tilde{k}_d$ is the following dimensionless mistuning $\tilde{k}_d=l_\varepsilon(k_1-k_2)$, and the dimensionless central wave number is given by

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$\tilde{k}_s = 2^{-1} l_\varepsilon (k_1 + k_2)$. Dimensionless correlation function of fluctuations $\tilde{A}(\zeta, \mathbf{r}) = l_\varepsilon^{-1} A(z, \mathbf{r})$ is expressed through the effective transversal correlation function of δ-correlated (in z-direction) fluctuations $A(z, \mathbf{r})$.

If taking account of the relationship between quantities K, $\tilde{k}_d$ and $\tilde{k}_s$: $\tilde{k}_s^2 = K + 4^{-1}\tilde{k}_d^2$, and re-scaling central transversal variable $\mathbf{R}$ according to $\left(K + 4^{-1}\tilde{k}_d^2\right)^{1/2} \boldsymbol{\rho} = \mathbf{R}$, then equation (1) is rewritten as

$$K\frac{\partial \Gamma_1}{\partial \varsigma} + \frac{i}{2}\left[\tilde{k}_d \nabla_{\mathbf{r}}^2 + \frac{\tilde{k}_d}{4}\left(K + \frac{\tilde{k}_d^2}{4}\right)^{-1} \nabla_\rho^2 - 2\nabla_\rho \nabla_{\mathbf{r}}\right]\Gamma_1 + \frac{K^2}{8} D(\varsigma, \mathbf{r})\Gamma_1 = 0 . \tag{2}$$

Here $D(\varsigma, \mathbf{r}) = 2\left[\tilde{A}(\varsigma, 0) - \tilde{A}(\varsigma, \mathbf{r})\right]$ is the effective transversal structure function of fluctuations.

The latter equation allows asymptotic solution at $K \to \infty$. Analysis of possible forms of the coherence function of the incident high frequency field indicates the following form of the solution

$$\Gamma_1\left(\mathbf{r}, \boldsymbol{\rho}, \varsigma, \tilde{k}_d, K\right) = \exp\left[K\psi\left(\mathbf{r}, \boldsymbol{\rho}, \zeta, \tilde{k}_d\right) + \sqrt{K}\Psi\left(\mathbf{r}, \boldsymbol{\rho}, \zeta, \tilde{k}_d\right)\right] \cdot \sum_{n=0}^{\infty} K^{-n/2} U_n\left(\mathbf{r},, \boldsymbol{\rho}, \zeta, \tilde{k}_d\right), \; n = 0, 1, \ldots \; . \tag{3}$$

Standard asymptotic procedure of substituting (3) into (2) results in the following “eikonal “ equations for “phase” functions ψ and Ψ

$$\partial\psi/\partial\zeta + i\tilde{k}_d (\nabla_{\mathbf{r}}\psi)^2/2 - i(\nabla_{\boldsymbol{\rho}}\psi \cdot \nabla_{\mathbf{r}}\psi) + D(\varsigma, \mathbf{r})/8 = 0, \tag{4}$$

$$\partial\Psi/\partial\varsigma + i\tilde{k}_d (\nabla_{\mathbf{r}}\psi \cdot \nabla_{\mathbf{r}}\Psi) - i(\nabla_\rho\psi \cdot \nabla_{\mathbf{r}}\Psi) - i(\nabla_r\psi \cdot \nabla_\rho\Psi) = 0 \tag{5}$$

and appropriate transport equations for amplitudes U_n. In order to save the space we only write here the main transport equation as follows

$$\begin{gathered}\frac{\partial U_0}{\partial \varsigma} + \frac{i\tilde{k}_d}{2}\left[2(\nabla_r U_0 \cdot \nabla_r \psi) + U_0 \nabla_r^2 \psi + U_0 (\nabla_r \Psi)^2\right] + \frac{i\tilde{k}_d^2}{8} U_0 (\nabla_\rho \psi)^2 - \\ i\left[(\nabla_\rho U_0 \cdot \nabla_r \psi) + (\nabla_r U_0 \cdot \nabla_\rho \psi) + U_0 \nabla_r \nabla_\rho \psi + U_0 (\nabla_r \Psi \cdot \nabla_\rho \Psi)\right] = 0.\end{gathered} \tag{6}$$

Complex trajectories correspond to the main eikonal equation (4) are described by the following set of the first order equations

$$\frac{d\varsigma}{d\tau} = 1, \; \frac{d\mathbf{r}}{d\tau} = i\tilde{k}_d \mathbf{p}_r - i\mathbf{p}_\rho, \; \frac{d\boldsymbol{\rho}}{d\tau} = -i\mathbf{p}_r \; , \tag{7}$$

$$\frac{dp_\varsigma}{d\tau} = -\frac{1}{8}\frac{\partial D(\varsigma, \mathbf{r})}{\partial \varsigma}, \; \frac{d\mathbf{p}_{\mathbf{r}}}{d\tau} = -\frac{1}{8}\nabla_r D(\varsigma, \mathbf{r}), \; \frac{d\mathbf{p}_\rho}{d\tau} = 0. \tag{8}$$

These equations define complex trajectories in 5-dimensional space $(\varsigma, \mathbf{r}, \boldsymbol{\rho})$. According to the first of equations (7) it can be accepted that the points along a trajectory are parameterised by a real parameter $\tau = \varsigma$. The trajectories arise from initial complex points $(\mathbf{r}_0, \boldsymbol{\rho}_0)$ at $\varsigma = 0$ and arrive at real points of observation $(\varsigma, \mathbf{r}, \boldsymbol{\rho})$, so that $(\mathbf{r}_0, \boldsymbol{\rho}_0)$ are the initial conditions to equations (7). In fact, $(\mathbf{r}_0, \boldsymbol{\rho}_0)$ are being determined as the function of the real point of observation $(\varsigma, \mathbf{r}, \boldsymbol{\rho})$ when the "inverse" homing problem is considered. Initial conditions to the moments $(\mathbf{p}_r, \mathbf{p}_\rho)$ from equations (8) are defined by a given initial distribution of the eikonal function $\psi_0(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d) = \psi(\mathbf{r}_0, \boldsymbol{\rho}_0, 0, \tilde{k}_d)$ of the coherence function of the incident field at $\varsigma = 0$ (see eq. (3)), so that

$$\mathbf{p}_{r0} = \nabla_r \psi_0, \ \ \mathbf{p}_{\rho 0} = \nabla_\rho \psi_0 . \tag{9}$$

As far as the initial condition to p_ς is concerned, it is obtained from equation (4) employing also relationships (9) at $\varsigma = 0$ as follows

$$p_\varsigma = -\frac{1}{2} i\tilde{k}_d (\nabla_r \psi_0)^2 + i(\nabla_{\boldsymbol{\rho}} \psi_0 \cdot \nabla_{\mathbf{r}} \psi_0) - \frac{1}{8} D(0, \mathbf{r}_0). \tag{10}$$

When dealing with the complex trajectories the analytical continuation of a structure function to the complex domain of its argument is an important point, and it should be specially discussed in each particular case. However, this is straightforward when the structure function of fluctuations is given explicitly by the analytic function. Then the appropriate Loran series can be written for different circles of convergence in the complex domain.

Once characteristic equations (7,8) with the initial conditions (9, 10) have been solved eikonal ψ from equation (4) is then obtained by the integral along the appropriate trajectory as follows

$$\psi(\mathbf{r}, \boldsymbol{\rho}, 0, \tilde{k}_d) = \psi_0(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d) + \int_0^\varsigma \left[p_\varsigma + i\tilde{k}_d \mathbf{p}_r^2 - 2i(\mathbf{p}_\rho \cdot \mathbf{p}_r) \right] d\varsigma . \tag{11}$$

When considering the second eikonal equation (5) for Ψ, it turns out that $d\Psi/d\varsigma = 0$ along the trajectories satisfying equations (7, 8) with the initial condition (9, 10), so that along any trajectory

$$\Psi(\mathbf{r}, \boldsymbol{\rho}, \zeta, \tilde{k}_d) = \Psi(\mathbf{r}_0, \boldsymbol{\rho}_0, 0, \tilde{k}_d) = \Psi_0(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d), \tag{12}$$

and $\Psi_0(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d)$ is defined by a given initial distribution of the second eikonal of the coherence function of the incident field at $\varsigma = 0$ (see eq. (3)). In particular, if this eikonal is not presented in the incident field $(\Psi_0 = 0)$, then it is also not present in the solution of the problem (3).

Finally, the solution to the main transport equation (6) is also given by integrating along the constructed trajectories as follows

$$U_0(\mathbf{r}, \boldsymbol{\rho}, \varsigma) = U_{00}(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d) \left\| \frac{\partial(\mathbf{r}, \boldsymbol{\rho})}{\partial(\mathbf{r}_0, \boldsymbol{\rho}_0)} \right\|^{-\frac{1}{2}} \exp\left[-\frac{i}{2} \int_0^\varsigma \left(\frac{\tilde{k}_d}{4} \mathbf{p}_\rho^2 + \tilde{k}_d^2 (\nabla_r \Psi)^2 - (\nabla_r \Psi \cdot \nabla_\rho \Psi) \right) \right] d\varsigma . \tag{13}$$

In equation (13) $U_{00}(\mathbf{r}_0, \boldsymbol{\rho}_0, \tilde{k}_d)$ is the distribution of the "amplitude" of the coherence function of the incident field, and the quantity under the sign of the square root is the determinant of the appropriate matrix of the derivatives of the points at a trajectory in the initial conditions to this trajectory.

SPACED POSITION AND FREQUENCY COHERENCE FUNCTION TO THE SPHERICAL WAVE

To demonstrate how the outlined technique works, when an incident field is a spherical wave written in the small-angle approximation respectively z-direction, its coherence function is

$$\Gamma_0\left(\mathbf{r}_0, \boldsymbol{\rho}_0, K, \widetilde{k}_d\right) = U_{00}\left(\mathbf{r}_0, \boldsymbol{\rho}_0, \widetilde{k}_d\right)\exp\left[K\psi\left(\mathbf{r}_0, \boldsymbol{\rho}_0, \widetilde{k}_d\right)\right] \tag{14}$$

with

$$\psi_0\left(\mathbf{r}_0, \boldsymbol{\rho}_0, \widetilde{k}_d\right) = i\boldsymbol{\rho}_0\left(\mathbf{r} + \widetilde{k}_d\boldsymbol{\rho}_0/2\right)/\varsigma_0, \quad U_{00}\left(\mathbf{r}_0, \boldsymbol{\rho}_0, \widetilde{k}_d\right) = \varsigma_0^{-2}\exp\left[i\widetilde{k}_d\varsigma_0^{-1}\left(\widetilde{k}_d\mathbf{r}\cdot\boldsymbol{\rho}_0 + \widetilde{k}_d^2\boldsymbol{\rho}_0^2/2 + \mathbf{r}^2/2\right)/4\right].$$

Here ς_0 is the dimensionless distance from the source of the spherical wave to the plane $\varsigma = 0$. According to (14) $\Psi_0 = 0$ in the coherence function of the incident spherical wave, so that eikonal Ψ at $\sqrt{K}$ in the general form of the solution (3) is not present in this case.

If also the statistical homogeneity of fluctuations along ς-direction is accepted, the outlines above general analysis can be performed analytically for some models of the structure function of fluctuations. In particular, if the simplest model of the structure function of fluctuations is accepted as

$$D(r) = 2\sigma^2 r^2\left(1 + r^2\right)^{-1}, \tag{15}$$

which tends to a positive constant $2\sigma^2$ as $r \to \infty$, analytical continuation into the complex domain of its argument is straightforward. In particular, its Loran series in the circle $|r| < 1$ and in the circle $|r| > 1$ are respectively

$$D(r) = 2\sigma^2 r^2\left(1 - \ldots\right) \quad \text{and} \quad D(r) = 2\sigma^2\left(1 - r^{-2} + \ldots\right). \tag{16}$$

When performing calculations according to presented here technique employing first of representations (16), complex trajectories are constructed which are launched and landed at $|r|, |r_0| < 1$, and the solution for the coherence function is constructed for small $|r|, |r_0|$. In this case it is equivalent to consideration of a problem of a spherical wave propagating in the medium with the quadratic structure function of fluctuations [4], and the explicit results are produced. In particular, in the limiting case of $\varsigma_0 \to \infty$ rigorous result to the plane wave propagating in a medium with the quadratic structure function of fluctuations [1] is obtained in an automatic fashion.

However, the most interesting is the case with trajectories located at $|r|, |r_0| > 1$, when the structure function of fluctuations is given by the second of representations (16). This is the case of realistic behaviour of the structure function of fluctuations at large values of its spaced position r. The explicit representation for the coherence function obtained employing the outlined technique for the second model in (16) demonstrates a proper behaviour (tending to a positive constant rather than to zero for unrealistic quadratic model) at large spaced position r.

REFERENCES

[1] I.Sreenivasiah, A.Ishimaru, and S.T.Hong, "Two-frequency mutual coherence function and pulse propagation in a random medium: An analytic solution to the plane wave case", *Radio Science,* vol. 11, pp. 775-778, 1976.
[2] J.Oz, and E.Heyman, "Modal solution to the plane wave two-frequency mutual coherence function in random media", *Radio Science,* vol. 31, pp. 1907-1917, 1996.
[3] A.Ishimaru, *Wave Propagation and Scattering in Random Media*, vols 1 and 2. New York: Academic, 1978.
[4] D.L.Knepp, "Analytic solution for the two-frequency mutual coherence function for spherical wave propagation", *Radio Science,* vol. 18, pp. 535-549, 1983.