INTEGRABILITY OF MAXWELL-BLOCH SYSTEMS

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ABSTRACT

A general mathematical framework for N-level Maxwell-Bloch systems is described. This model differs from standard models for 2-level systems in a number of respects which are quite significant for the search for more general integrable models of nonlinear electrodynamics. The fundamental formulation is in terms of the vector potential coupling to the atomic dipoles. The full time-domain behaviour of the wave is retained throughout, there being no decomposition into carrier and envelope parts with an associated assumption of different timescales for the two. The integrability of these systems depends on a symmetry under automorphism of an underlying Lie algebra.

INTRODUCTION

Maxwell-Bloch (MB) systems are generically composed of coupled systems of Maxwell equations and fundamental quantum-mechanical dynamical systems. They describe electromagnetic waves in general atomic media, including the effects of dispersion, dissipation, external pumping, active and passive nonlinearity and even resonance. Usually, they have been studied under restrictions to 1-dimensional wave propagation (plane waves) and simplified quantum models such as an ensemble of independent 2-level atoms, but are sufficiently generic to model a wide class of phenomena where electromagnetic waves interact with a material medium.

The development of mathematical techniques of solution of MB systems is almost completely unexplored beyond the case of 1-dimensional wave + 2-level atoms, despite the universality of this description of important electromagnetic wave phenomena. Even so, it is known that even the simplest of these models may be exactly integrable under certain conditions which include resonance of the wave with atomic transitions of the medium, but which neglect dissipation. These resonant models include the phenomenon of self-induced transparency (SIT) [1]. Under off-resonance conditions, after extracting the rapidly oscillating carrier and approximating the off-resonant atomic dynamics by multiple-scale asymptotic techniques, elementary MB systems reduce to Nonlinear Schrodinger Equations (NSE) for the wave envelope; the NSE is similarly known to be completely integrable, with a particularly extensive literature.

Nonlinear wave systems of the completely integrable type are of particular interest because their solutions can be constructed explicitly for arbitrary initial conditions using the well-known Inverse Scattering Transform. Integrable systems may exhibit soliton phenomena, which are of particular importance in the study of ultrafast (sub-picosecond) interactions in nonlinear optics. Solitons can be generated mathematically by means of Backlund Transformations.

In this paper a general mathematical framework for N-level Maxwell-Bloch systems will be described, based on the structure of integrable models which are already known. This model differs from standard models for 2-level systems in a number of respects which are quite significant for the search for more general integrable models of nonlinear electrodynamics. First, the fundamental formulation is in terms of the vector potential coupling to the atomic dipoles, so-called minimal replacement coupling, rather than the conventional electric dipole coupling that has been universally used in the 2-level cases. It has already been noted elsewhere that the nonlinear optics of nonresonant finite-dimensional quantum systems may differ quantitatively depending on whether the vector potential or electric field defines the coupling [2]. A further difference between the new model and most classical 2-level models is the fact that the full time-domain behaviour of the wave is retained throughout, there being no decomposition into carrier and envelope parts with an associated assumption of different timescales for the two. Finally, the integrability of these systems depends on a symmetry under automorphism of an underlying Lie algebra, a subtle and elegant mathematical feature which is completely new in the treatment of electrodynamical

problems. Some new cases of completely integrable systems exhibiting electromagnetic wave temporal solitons are presented, including the case of arbitrary elliptical optical polarisation, so-called vector solitons.

N-LEVEL MAXWELL-BLOCH SYSTEMS

The equations governing a spatially homogeneous ensemble of N-level quantum systems (bound electrons in atoms) interacting with a plane electromagnetic wave consist of the Liouville equation for the dynamics of the quantum density matrix, and Maxwell's equations for the electromagnetic field

$$\mathrm{i}\hbar\partial_t\Gamma = [H,\Gamma], \tag{1}$$

$$\partial_z^2\mathbf{A} - c^{-2}\partial_t^2\mathbf{A} = -c^{-2}\epsilon_0^{-1}\partial_t\mathbf{P} \tag{2}$$

where $\mathbf{A}(z,t)$ is the vector potential field of the plane wave, assumed to be transverse to the direction of propagation, $\mathbf{P}(z,t)$ is the polarisation density induced in the atomic ensemble, c is the velocity of light *in vacuo*, Γ is the $N\times N$ density matrix, and H is the $N\times N$ Hamiltonian matrix given by

$$H = H_0 + \frac{e}{m}\mathbf{p}\cdot\mathbf{A} \tag{3}$$

where H_0 is the diagonal matrix of electron eigenenergies, $-e$ and m are the charge and mass of the electron, respectively, and $\mathbf{p}$ is the momentum matrix. The polarisation density $\mathbf{P}$ is given by

$$\mathbf{P} = -\tfrac{1}{2}Ne\mathrm{Tr}\{\Gamma\mathbf{Q}\}, \tag{4}$$

where N is the number density of atoms, and $\mathbf{Q}$ is the dipole displacement operator. The relation between the matrices $\mathbf{p}$ and $\mathbf{Q}$ is

$$\mathbf{p} = (\mathrm{i}\hbar)^{-1}m[\mathbf{Q},H_0]. \tag{5}$$

The only approximation to be introduced in these equations is that the density of atoms in the ensemble is small, in which case reflected waves can be neglected and the *one-way wave* approximation results

$$(\partial_z + c^{-1}\partial_t)\mathbf{A} = \frac{1}{2\epsilon_0 c}\mathbf{P}. \tag{6}$$

in place of (2). By substituting the shifted time variable $\tau = t - z/c$, equations (1) and (6) reduce to

$$\mathrm{i}\hbar\partial_\tau\Gamma = [H,\Gamma] \tag{7}$$

$$\partial_z\mathbf{A} = \frac{1}{2\epsilon_0 c}\mathbf{P}, \tag{8}$$

which are the *N-level reduced Maxwell-Bloch (RMB) equations*.

Several special cases of (7)–(8) are already known to be completely integrable. The earliest example of an integrable system based on (7)–(8) was given in [1], for $N=2$. Recently [3], a rotationally symmetric version with $N=3$ has been shown to be integrable with vector solitons. It has also been shown that the integrability is preserved when a uniform static axial magnetic field is applied, leading to Faraday rotation of the vector soliton.

The general strategy for investigating integrability depends on the reduction of the MB system to a *zero-curvature* form

$$\partial_z U - \partial_\tau V + [U,V] = 0 \tag{9}$$

for two matrices U and V which depend on a parameter ζ, so that (9) holds identically for *all* ζ. The parameter ζ may be introduced quite naturally because the density matrix and the hamiltonian are elements of the Lie algebra su(N), and su(N) possesses inner automorphisms which preserve all its commutator relations. The general inner automorphism is $X \to -X^\dagger$, where X represents any element of su(N). In a basis of su(N) $X \in \mathrm{su}(N) = \mathrm{i}\sum_j X_j\lambda_j$ consisting of elements λ_j which are either pure real or pure imaginary and real coefficients X_j, the inner automorphism consists of changing the sign of all real λ_j, while leaving unchanged all

imaginary λ_j. There are $\frac{1}{2}N(N-1)$ imaginary off-diagonal basis elements, $\frac{1}{2}N(N-1)$ off-diagonal real basis elements, and $N-1$ real diagonal elements in the matrix representation of su(N). Under this automorphism, the density matrix and the Hamiltonian split into two components which are symmetric or antisymmetric:

$$\rho = \rho^+ + \zeta\rho^- \tag{10}$$
$$H = H^+ + \zeta H^- \tag{11}$$

$$\rho^- = \sum_{j=1}^{\frac{1}{2}N(N-1)} \rho_j\lambda_j, \quad \rho^+ = \sum_{j=\frac{1}{2}N(N-1)+1}^{N^2-1} \rho_j\lambda_j, \tag{12}$$

$$H^- = \sum_{j=1}^{\frac{1}{2}N(N-1)} H_j\lambda_j, \quad H^+ = \sum_{j=\frac{1}{2}N(N-1)+1}^{N^2-1} H_j\lambda_j. \tag{13}$$

With $\zeta = \pm 1$ the same sets of real numbers $\{\rho_j\}$ and $\{H_j\}$ satisfy the Liouville equation. Having identified the automorphism, the definitions of the density matrix and the Hamiltonian can be extended to all values of ζ, not just $\zeta = \pm 1$. In particular, we define matrices

$$U(\zeta) = (\mathrm{i}\hbar)^{-1}(H^+ + \zeta H^-), \tag{14}$$
$$V(\zeta) = \mathrm{i}\sum_k \frac{K_k}{\zeta^2 - \zeta_k^2}(\rho_k^+ + \zeta\rho_k^-) \tag{15}$$

with some arbitrarily chosen poles $\pm\zeta_k$ and amplitudes K_k in the definition of $V(\zeta)$. For these two matrices to satisfy the zero-curvature condition (9) identically, it is required that

$$\partial_z H^+ = -\mathrm{i}\sum_{k=1}^{K} K_k[H^-, \rho_k^-] \tag{16}$$
$$\partial_z H^- = 0 \tag{17}$$
$$\mathrm{i}\hbar\partial_\tau(\rho_k^+ \pm \zeta_k\rho_k^-) = [(H^+ \pm \zeta_k H^-), (\rho_k^+ \pm \zeta_k\rho_k^-)] \tag{18}$$

Equation (18) represents a set of Liouville equations for density matrices ρ_k and Hamiltonians $H_k = H^+ + \zeta_k H^-$. Each Hamiltonian H_k corresponds to an atom whose energy eigenvalues are all scaled by the same factor ζ_k but are otherwise identical; this is a model for inhomogeneous broadening of the resonant optical transition. In all cases arising from optical interactions, the antisymmetric component of the Hamiltonian H^- is constant, and so (17) is automatically satisfied. For the symmetric component H^+, it is noted that the momentum matrix elements are pure imaginary, so this component is just $H^+ = (e/m)\mathbf{p}\cdot\mathbf{A} = (e/m)\sum_{j=x,y} p_j A_j \sigma_j$, where $\sigma_j = \sum_{k=1}^{N} c_{jk}\lambda_k$ is the unit momentum matrix for dipole coupling along the cartesian axis $j = x, y$. The elements of H^- are pure real diagonal, and therefore H^- commutes with the diagonal elements of ρ_k^-, which may consequently be ignored in the commutator on the right of (16). The final reduction of (16) is obtained by projecting both sides onto the dipole matrices σ_j, using the Lie algebra inner product $\frac{1}{2}\mathrm{Tr}\{\sigma_i\sigma_j\} = \delta_{ij}$:

$$\frac{ep_j}{m}\partial_z A_j = \tfrac{1}{2}\mathrm{i}\sum_{k=1}^{K} K_k \mathrm{Tr}\{\rho_k^-[H^-, \sigma_j]\} \tag{19}$$

which, after multiplying both sides by p_j and using (5), reduces to

$$ep_j^2\partial_z A_j = -\tfrac{1}{2}\hbar^{-1}m^2\sum_{k=1}^{K} K_k \mathrm{Tr}\{\rho_k^-[H^-, [H^-, Q_j]]\}. \tag{20}$$

Usually for optical Hamiltonians $[H^-, [H^-, Q_j] = \gamma_j Q_j$ with real positive constants γ_j, so

$$\partial_z A_j = -\frac{m^2\gamma_j}{e\hbar p_j^2}\sum_{k=1}^{K} K_k\langle Q_j\rangle_k \tag{21}$$

where $\langle Q_j\rangle_k = \frac{1}{2}\mathrm{Tr}\{\rho_k^- Q_j\}$. Equation (21) is similar to the Maxwell equation for coupling of an electromagnetic field to an anisotropic homogeneous ensemble of K different kinds of atoms such that the macroscopic polarisation density induced by the mixture is proportional to the right-hand side. However, the normal appearance of the one-way wave equation does not have the orientation-dependent constants preceding the macroscopic average on the right-hand side. These constants only become independent of the axis j when certain *symmetry* conditions are present.

Generally there are more elements in H^+, $\frac{1}{2}N(N-1)$ in number, than there are cartesian coordinates, 2 in number, for the transverse field $\mathbf{A}$. Hence there are $\frac{1}{2}N(N-1)-2$ constraints implied by (16):

$$0 \quad = \quad \tfrac{1}{2}\mathrm{i}\sum_{k=1}^{K} K_k \mathrm{Tr}\{\rho_k^-[H^-,\sigma_j]\}, \quad 3 \le j \le \tfrac{1}{2}N(N-1), \tag{22}$$

where it is assumed that $\{\sigma_j : 1 \le j \le \frac{1}{2}N(N-1)$ form an orthonormal basis for the symmetric part of the Lie algebra, with σ_1 and σ_2 chosen to represent the momentum along x and y respectively. These constraints severely restrict the systems integrable by this method. For example, for $N=3$ the constraint $[H^-,\sigma_3]=0$ means that the two excited energy levels in the 3-level system must have equal energies, *i.e.* they are *degenerate*. When the energy levels are degenerate, the coefficients $\gamma_{1,2}$ are identical. For an isotropic medium, the p_j and γ_j are independent of j: $p_{1,2}=p$ and $\gamma_{1,2}=\gamma$. In this case the system reduces to

$$\partial_z A_j \quad = \quad -\frac{e}{2c\epsilon_0}\sum_{k=1}^{K} N_k \langle Q_j\rangle_k \tag{23}$$

with the choice of

$$K_k = \frac{N_k \hbar p^2 e^2}{2m^2\gamma c\epsilon_0} \tag{24}$$

with N_k the number density of the k-th atomic species. Equation (23) is the correct Maxwell equation for the potential coupled to two orthogonal dipoles with rotational symmetry.

CONCLUSIONS

A method of constructing nonlinear integrable systems from combinations of the one-way Maxwell wave equation and simple N-level quantum-electronic systems has been described, with an attempt to gain some generality. Certain properties of Lie algebras, such as inner automorphisms, are fairly central to the theory, and this is the first time that such tools have been required in electrodynamics. The system (23) describes one-way propagation of a polarised electromagnetic wave in an isotropic inhomogeneously broadened atomic medium, with rotational symmetry about the z-axis. The inhomogeneous broadening is described by the variation of the diagonal part of the Hamiltonian by the factor ζ_k, with the coefficients N_k representing the number density of the k-th component of the mixture. Inhomogeneous broadening is inessential for the integrability of the system, this property being preserved in the case $K=1$. Despite the constraints introduced to make the integrable system correspond to a physical system, the integrable system has a wide range of realisations in physical media.

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