Finite Element Solution for Magnetostatics using Vector Magnetic Potential

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Abstract

This paper describes the derivation of the equations necessary to solve magnetostatics problems by the finite element method using the vector-magnetic potential with focus on the incorporation of boundary conditions. The vector-magnetic potential is not a real quantity that can be measured, but a factor used to simplify the computation required to obtain the magnetic field. From the definition of a functional, the process of the derivation is outlined along with descriptions of the approximating functions. In two dimensions, the excitation current and the magnetic field is typically assumed to be independent of the longitudinal dimension, which is the *z*-direction. To validate this assumption, a mathematical proof is provided for the nonexistence of a linear *z* directed behavior in the vector-magnetic potential. This allows the magnetic field to be considered only in the transverse or *x-y* plane when the source only has a *z*-directed component, which is usually done in practice. The important contribution of this work is the inclusion of the boundary conditions within the functional formulation.

Introduction

The vector potential is treated as a continuous function and incorporation of discontinuities at material boundaries and interfaces does not require any special considerations. The development of a solution to Maxwell's equations for magnetostatics can start with a variety of methods such as the Galerkin weighted residual function. Due to the existence of a functional, which guarantees a minima, a variational approach is chosen. The incorporation of the boundary conditions and the effects of different boundary conditions on the solution is the primary focus of the derivation and this paper.

The Functional

The required equations are $\nabla\times\overline{H}=\overline{J}$ and $\nabla\cdot\overline{B}=0$. The latter allows us to write $\overline{B}=\mu\overline{H}=\nabla\times\overline{A}$. Thus, we may write an equation for the vector potential as

$$\nabla\times\left(\frac{1}{\mu}\nabla\times\overline{A}\right)-\overline{J}=0$$

with the boundary conditions as constraints. Two types of typical boundary conditions are the Dirichlet type, $\hat{n}\times\overline{A}$, and the Neuman type, $\hat{n}\times\frac{1}{\mu}\nabla\times\overline{A}$. The vector $\overline{A}$ represents the vector-magnetic potential, $\overline{J}$ the current source term, and $\hat{n}$ represents the outward normal to the boundary. The functional was developed using Lagrange multipliers with the final result given by

$$I(\overline{A}) = \int_{\Omega} \left(-\frac{1}{\mu} \left(\nabla \times \overline{A} \right) \cdot \left(\nabla \times \overline{A} \right) + 2\overline{A} \cdot \overline{J} \right) d\Omega + 2 \int_{\Gamma_N} \left(\overline{A} \times \overline{H}_b \right) \cdot d\overline{\Gamma}$$

$\overline{H}_b$ is the magnetic field at the Neuman boundary. In the finite element formulation, the Dirichlet boundary value is enforced directly and not treated as an unknown. This functional can be interpreted as a solution which takes into account the total stored energy, the source current, and the boundary current. The Neuman boundary condition can also interpreted as a surface current $\overline{J}_s$,

$$\overline{J}_s = \hat{n} \times \overline{H}_b$$

When this surface current is zero, the problem is said to have a natural boundary. The above equation is a result of an extrema and the derivation of the functional to this point has included both the Dirichlet and Neuman boundary conditions. It is seen that the Neuman boundary condition remains, while the Dirichlet condition is eliminated from the functional and has to be imposed, if required, directly on the poitential. The functional accounts for the energy in the system, using Maxwell's equation as a constraint and incorporating the required boundary conditions.

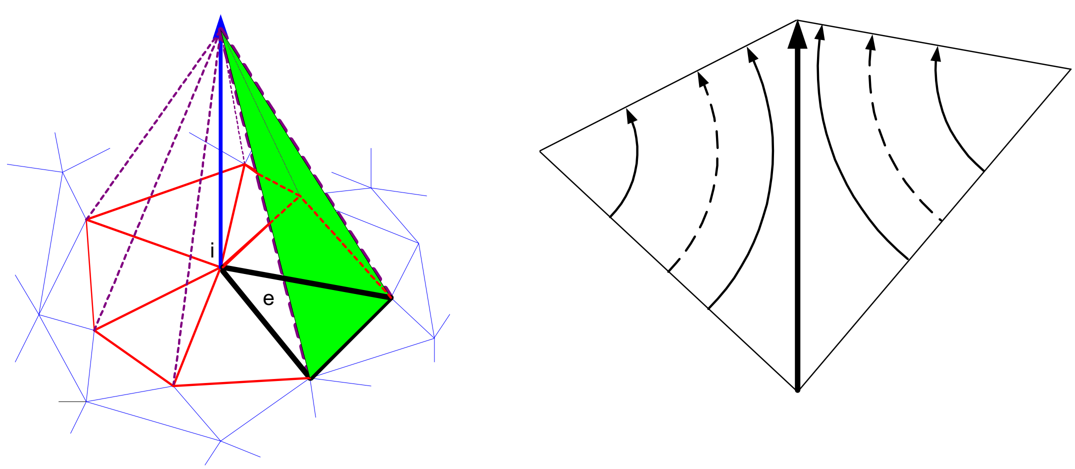

Figure 1. Node and tangential basis for the 2-D problem.

Nodal and Tangential-edge Bases

Nodal basis functions are typically expressed as a pyramid bases that have a value of unity at the node under consideration and linearly fall to zero at all surrounding nodes. Node *i* is an internal node and the basis function Φ_i in element e is highlighted in Fig. 1.

Tangential edge based basis have a constant tangential component along each edge. Fig. 1 shows the pyramid type of behavior in a typical triangular mesh and the linear variation of edge basis for a common edge in two elements.

Lifting Electromagnet

The first example demonstrates the ability of the vector-magnetic potential method to provide an accurate solution when the domain of interest contains different materials. Fig. 2 shows a lifting electromagnet magnet in a vacuum and a steel bar as the load a short distance away. The coils are also assumed to have the same permeability as that of vacuum and carry current only in the +/- *z* directions.

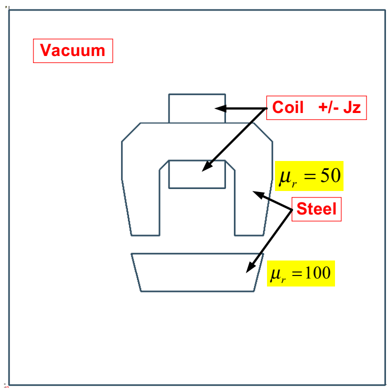

Figure 2. Electromagnet

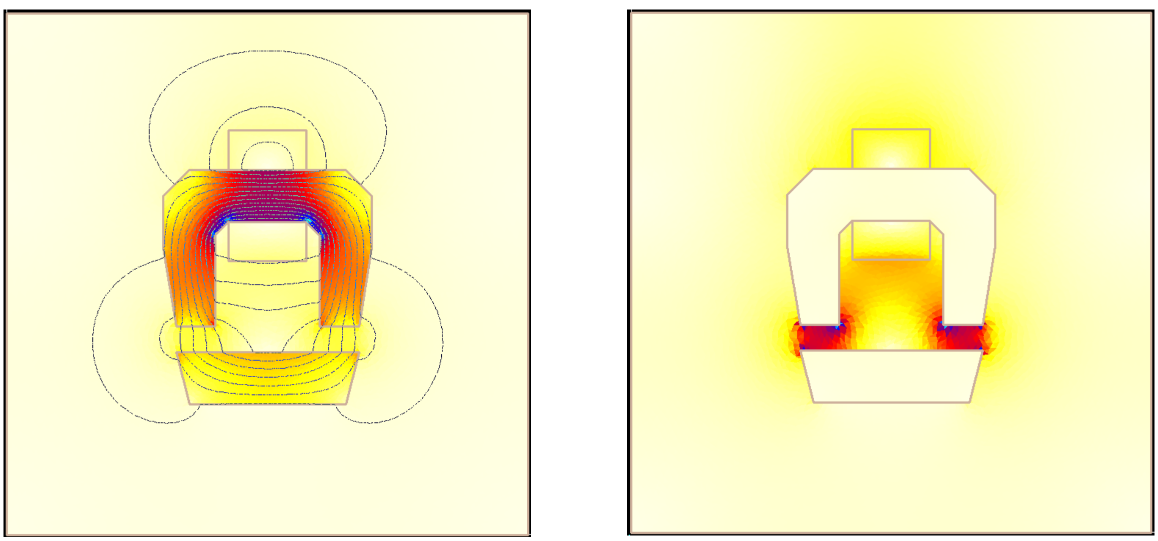

Figure 3. $|B_t|$ and $|H_t|$ for electromagnet

The magnetic flux density is obtained by taking the curl of the potential and is shown in Fig. 3a, it is piecewise constant and is shown with a constant value in each element. The contours of the magnetic potential shows the actual paths of the magnetic flux. Fig. 3b shows the magnetic field which is also piecewise constant. Note the value of near zero in the steel and the dominance in the gap between the steel pieces. Leakage paths in the vicinity of the magnet can also be observed. The steel is assumed to be linear and the maximum flux density is approximately 1.0 T. Also, the Dirichlet boundary condition with a value of 0 was applied to the entire boundary. That is, the magnetic potential at the four sides of the boundary was set to zero. Nodal basis functions were used for the above problem. The color scale is from bright yellow to dark blue with increasing intensity due to the ease of viewing.

Conclusion

Applications of node-based finite elements have been described. The node based finite element solution allows the excitation current to have a constant magnitude in all or any of the three directions. This problem can be reduced to two separate and independent problems, the solution presented here is with the source components in the *z* direction. Additional examples and edge based solutions will be presented at the conference.

The requirement of a gauge and the required boundary conditions for uniqueness of the solution will also be considered. Several results highlighting solutions with excitations that are *z* directed as well as those in the *x-y* plane will be presented, with the domain of interest subdivided with Delaunay triangulation. The effect on the solution when different boundary conditions are enforced will be shown. Results obtained by using several different solution algorithms (LU decomposition, sparse conjugate gradient and sparse bi-conjugate gradient methods) may be briefly compared with respect to CPU time, memory usage and accuracy of the results. The material presented in this paper partially forms the background required to find solutions to both static and dynamic problems in electromagnetics.