THE DIFFRACTION BY A CURVED IMPEDANCE WEDGE : DIFFRACTED AND CREEPING WAVES.

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The methods generally considered for the diffraction by a wedge with curved faces are often different as you consider a wedge of narrow or large angle. An original approach, initiated in [1] for faces of constant curvatures and whose results have been recently used by Borovikov in [2] for his developments on non constant curvature case, permits a new global asymptotic study for an impedance curved wedge of any angle, and for points far from or close to the curved surface. The expressions presented comprise high order asymptotic terms, which allows to consider simultaneously diffracted phenomena appearing at different orders.

In this approach, we reduce the boundary problem on the field to an asymptotic one on the spectral function which is related to the representation of the field in the form of a Sommerfeld-Maliuzhinets integral. The functional equations then obtained can be solved, and the asymptotics of the spectral function is developed [1]. A new form of it is presented here. An expression allowing the transition to an asymptotic expansion with fractional powers is also given for directions close to the tangent at the faces, where creeping waves excited by the edge propagate. We note that the general expressions here given allow to recover all known results on diffraction coefficients for particular wedge angles, from the case of a discontinuity of curvature [3] to the case of a curved half plane [4].

1) Asymptotic expressions with no creeping waves terms

We consider a wedge with tangent planes at the edge defined by $\varphi = \pm\Phi$, in cylindrical coordinates (ρ,φ), and curved faces of radius $a^{\pm}$, with $\Phi > \pi/2$. The electric and magnetic potentials E_z and H_z, indifferently denoted u, verify outside the wedge $(\Delta + k^2)u = 0$, with $k = \omega(\epsilon\mu)^{1/2}$ the wave number of the free space and $e^{i\omega t}$ time dependence. They are represented, for plane wave illumination normal to the edge, in the form of an integral of Sommerfeld-Maliuzhinets with a contour γ composed of two infinite loops in the complex plane [1], for $|\varphi| < \Phi$,

$$u(\rho,\varphi) = \frac{1}{2\pi i}\int_{\gamma} f(\alpha+\varphi)e^{ik\rho\cos\alpha}d\alpha\,, \qquad (1)$$

with f to be defined. After deformation of γ to the steepest descent path SDP, we can write

$$u(\rho,\varphi) = u_i + \sum u_r^{\pm} + \sum u_s^{\pm} + \frac{\mathrm{e}^{-ik\rho}}{2\pi i}\int_{SDP} f(\alpha+\varphi)\,\mathrm{e}^{ik\rho(\cos\alpha+1)}\,\mathrm{d}\alpha, \quad (2)$$

where

- the term u_i is the *incident field* in the illuminated zone and *zero* in the shadow zone;
- the terms $u_r^{\pm}$ correspond to geometrical optics *field reflected* by each faces $\varphi = \pm\Phi$ (note: some secular terms are present in the asymptotic development of this term at high order).
- the terms $u_s^{\pm}$ are terms of *guided waves excited by the edge,* which are the contribution of complex pôles $\alpha_s^{\pm}$ of $f(\alpha+\varphi)$;
- the last term, named u_d, *is principally radiated cylindrically when* $\rho\to\infty$. Approximating $f(\alpha+\varphi)$ on $SDP_{\pm}$ by its value at the stationnary phase points $\alpha = \pm\pi$, we obtain

$$u_d = \frac{-e^{-i\pi/4-ik\rho}}{\sqrt{2\pi k\rho}}[f(\pi+\varphi)-f(-\pi+\varphi)]+O(1/(k\rho)^{3/2}) \qquad (3)$$

where $F(\varphi) = f(\pi+\varphi)-f(-\pi+\varphi)$ is the *diffraction (or far field) coefficient.*

We assume impedance boundary conditions on the surface of the wedge,

$$(\frac{\partial}{\partial n} - ik\sin\theta^{\pm})u|_{curved\,face\pm} = 0 \qquad (4)$$

for n the outward normal, $\sin\theta^{\pm}$ a constant. We approximate (4) by *asymptotic conditions* in entire powers of the curvatures $1/a^{\pm}$, on tangent planes at the edge $\varphi = \pm\Phi$ [1,5] :

$$(\mp\frac{\partial}{\rho\partial\varphi} - ik\sin\theta^{\pm} - \frac{1}{2a^{\pm}}((\frac{\partial^2}{\partial\varphi^2} - \rho\frac{\partial}{\partial\rho}) \pm ik\sin\theta^{\pm}\frac{\rho\partial}{\partial\varphi}) + O(\frac{\rho}{(a^{\pm})^2}))\, u\,|_{\varphi=\pm\Phi} = 0.$$

We can show, by applying the Maliuzhinets inversion theorem, that we obtain *the asymptotic functional equation,*

$$(\sin\alpha \pm \sin\theta^{\pm} \pm \frac{1}{ika^{\pm}}D^{\pm}_{\alpha}(\,.\,) + O(\frac{1}{(ka^{\pm})^2}))f(\alpha\pm\Phi) \qquad (5)$$
$$-(-\sin\alpha \pm \sin\theta^{\pm} \pm \frac{1}{ika^{\pm}}D^{\pm}_{-\alpha}(\,.\,) + O(\frac{1}{(ka^{\pm})^2}))f(-\alpha\pm\Phi) = 0\,,$$

where

$$D^{\pm}_{\alpha}(.) = \frac{1}{2}\,[\,\frac{\partial^2(.)}{\partial\alpha^2} - \frac{\partial}{\partial\alpha}(\mathrm{cotg}\alpha(.)) \pm \sin\theta^{\pm}\frac{\partial}{\partial\alpha}(\frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}(.))\,]\,. \qquad (6)$$

We then let $f = \sum_{n\geq 0} f_n/k^n$ in the functional equation. Considering the terms of same powers in k, we obtain a succession of functional equations of order m on the $f_{n\leq m}$, that we solve. The expression of $f_\circ$ corresponds to the spectral function for a wedge with plane faces [6].

Let us develop the integral expression given initially in [1] for f_1, *the first term influenced by the curvatures*. For this, we let $f_\circ(\alpha) = \Psi(\alpha)\sigma(\alpha)$ where $\sigma(\alpha) = \mu'\cos(\mu'\varphi_\circ)/(\sin(\mu'\alpha) - \sin(\mu'\varphi_\circ))$, $\mu' = \pi/2\Phi$, $\varphi_\circ$ the angle of incidence, and $f_1(\alpha)/k = \Psi(\alpha)\chi(\alpha)/k$. Considering the initial expression of χ in [1], we shift the integration path with the real quantities $\pm d^{\pm}$ along the real axis, for each term concerning the faces $\pm$, and we obtain the following expression :

$$\chi(\alpha') = \frac{i}{8\Phi\Psi(\varphi_\circ)}\sum_{\pm}\frac{1}{ia^{\pm}}(\int_{-i\infty\pm d^{\pm}}^{i\infty\pm d^{\pm}} d\alpha[(-W^{\pm}\partial_\alpha\ln((-\sin\alpha\pm\sin\theta^{\pm})\Psi(-\alpha\pm\Phi))$$
$$+\partial_\alpha W^{\pm})(-\partial_\alpha\ln(\frac{\Psi(-\alpha\pm\Phi)}{(\sin\alpha\pm\sin\theta^{\pm})})\,\sigma(\alpha\pm\Phi) - \partial_\alpha\sigma(\alpha\pm\Phi))\,\frac{1}{\sin\alpha}\,]$$
$$\mp 2\pi i\ \mathrm{sgn}d^{\pm}.\sum_{s}\mathrm{Residu}[..]\,|_{\alpha=\alpha'_s\ pole\ of\,[..]\quad 0<\pm\mathrm{sgn}d^{\pm}Re\alpha'_s<|d^{\pm}|}\ ) \qquad (7)$$

where $W^{\pm} = \mathrm{tg}(\frac{\pi}{4\Phi}(\alpha'\pm\Phi-\alpha))$, Ψ is defined in [6] for wedge with plane faces, and the constants $d^{\pm}$ are arbitrary. We note that this expression is the *analytical continuation* of the one in [1] for $\alpha' \in]-3\Phi+d^{+},\Phi+d^{+}[\,\cap\,]-\Phi-d^{-},3\Phi-d^{-}[$.

2) Asymptotic expressions with creeping waves terms

We note, from (5) and residu terms in (7), that $f_1(\pm\pi+\varphi)$ and then the expression (3) of u_d are singular for $\varphi = \pm\Phi$. In fact, near the tangents of the faces, it is preferable to rewrite the radiation function F, from a transformation of the previous asymptotics of F (at any order), following [1,5]

$$F(\varphi) \equiv \frac{-\,\mathrm{e}^{ika^{\pm}\sin(\pm\varphi-\Phi)}}{i\pi ka^{\pm}} \int_{-\infty-i\epsilon_1}^{+\infty-i\epsilon_1} \frac{\mathrm{e}^{-i\nu(\pm\varphi-\Phi)}\mathcal{M}^{\pm}(\pm\nu)}{H_{\nu}^{(2)\prime}(ka^{\pm})+Q^{\pm}\,H_{\nu}^{(2)}(ka^{\pm})}\,\mathrm{d}\nu, \quad (8)$$

when $|\pm\varphi-\Phi| < \pi/2$, with $Q^{\pm} = -\,i\sin\theta^{\pm}$, and

$$\mathcal{M}^{\pm}(ka^{\pm}\sin\alpha) = \frac{i\pi ka^{\pm}}{2\cos\alpha}\,.\mathcal{O}_{\alpha}^{\pm}(F(\alpha\mp(\pi/2-\Phi))) \qquad (9)$$

where $\mathcal{O}_{\alpha}^{\pm}$ is an asymptotic differential operator, that can be deduced at any order from the asymptotic boundary conditions considered on each tangent plane $\varphi = \pm\Phi$, with

$$\begin{aligned}\mathcal{M}^{\pm}(\pm ka^{\pm}\cos\alpha) &= -\,i\pi ka^{\pm}\,\mathcal{O}_{\pm(\alpha+\pi/2)}^{\pm}(F_{<}(\pm(\alpha+\Phi)))/2\sin\alpha\\ &= \frac{1}{\sin\alpha}([(\sin\alpha-\sin\theta^{\pm}).(\,.\,)-D_{\pm\alpha+\pi}^{\pm}(\,.\,)/ika^{\pm}+0(1/(ka^{\pm})^2)]\,F(\pm(\alpha+\Phi)). \quad (10)\end{aligned}$$

Thereafter, we note $\mathcal{A}_{\alpha} = -\,\mathcal{O}_{\pm(\alpha+\pi/2)}^{\pm}/(2\sin\alpha)$.

The expression (8) of F is uniform in the region $\varphi\sim\pm\Phi$. It permits the *continuous* transformation of the field into creeping waves (a development in entire powers then changes continuously to one in fractional powers as φ varies), for any wedge angle.

We can derive, from (8), an expression at finite distance for observation point (ρ',φ') and point source (ρ,φ) radiating $u^i = \frac{i}{2}H_{\circ}^{(2)}(k\sqrt{\rho^2+(\rho')^2+2\rho\rho'\cos(\varphi-\varphi')})$. For this, we let $\mathcal{L}^{\pm}(\pm ka^{\pm}\cos\alpha') = \mathcal{A}_{\alpha'}(\frac{1}{2\pi i}\int_{\gamma} f(\alpha+\varphi,\pm(\alpha'+\Phi))e^{ik\rho\cos\alpha}d\alpha)$, and obtain, when $\varphi'\sim\pm\Phi$,

$$u(\rho',\varphi',\rho,\varphi) \;\equiv\; \frac{1}{2i}\int_{\mathcal{E}} \frac{\mathcal{L}^{\pm}(\pm\nu')\,H_{\nu'}^{(2)}(kR')\,e^{\mp i\nu'\Omega'}}{H_{\nu'}^{(2)\prime}(ka^{\pm})+Q^{\pm}H_{\nu'}^{(2)}(ka^{\pm})}\,d\nu' \qquad (11)$$

for $ka^{\pm}$ large, where $R' = \sqrt{\rho'^2+(a^{\pm})^2+2\rho' a^{\pm}\cos(\Phi-\pi/2\mp\varphi')}$ and Ω' is the angle from the edge to the observation point, at $\pm$-curvature center. When we go beyond the frontier $\varphi' = \pm\Phi$, we can consider the residus terms at zeros $\nu_p^{\pm} = ka^{\pm}\cos\alpha_p^{\pm}$ of $[H_{\nu'}^{(2)\prime}(ka^{\pm})+Q^{\pm}H_{\nu'}^{(2)}(ka^{\pm})]$ in the zone $Im\nu < 0$, closing the contour $\mathcal{E}$ at infinity. We then obtain when $|\varphi| < \Phi$ and $|\varphi'| > \Phi$,

$$u(\rho',\varphi',\rho,\varphi) = -\,\pi\sum_{\nu_p}\mathcal{L}^{\pm}(\pm\nu_p^{\pm})\,H_{\nu_p}^{(2)}(kR')e^{\mp i\nu_p\Omega'}\Big/\frac{\partial}{\partial\nu}(H_{\nu'}^{(2)\prime}(ka^{\pm})+Q^{\pm}H_{\nu'}^{(2)}(ka^{\pm}))|_{\nu'=\nu_p^{\pm}} \quad (12)$$

where $\mathcal{L}^{\pm}(\pm ka^{\pm}\cos\alpha_p) = \mathcal{A}_{\alpha_p}(\frac{1}{2\pi i}\int_{\gamma} f(\alpha+\varphi,\pm(\alpha_p+\Phi))e^{ik\rho\cos\alpha}d\alpha)$ can be developed as $\mathcal{A}_{\alpha_p}(u_{i,p}+\sum u_{r,p}^{\pm}+\sum u_{s,\epsilon,p}^{\pm}+\frac{\mathrm{e}^{-ik\rho}}{2\pi i}\int_{SDP} f(\alpha+\varphi,\,\pm(\alpha_p+\Phi))\,\mathrm{e}^{ik\rho(\cos\alpha+1)}\,\mathrm{d}\alpha$.

For $l\varphi > \Phi$ et $l'\varphi' > \Phi$ (l,l' are equal to $+$ ou $-\,1$), the problem is more complex and we need to express $\mathcal{L}^{\pm}(\pm\nu')$ as an integral of previous form. We then obtain for the diffracted part of the field :

$$\begin{aligned}u_{da} = \pi^2\sum_{p,q}\mathcal{A}_{\alpha_p^l}^{l}\mathcal{A}_{\alpha_q^{l'}}^{l'}(f(\pi+l(\alpha_p^l+\Phi),l'(\alpha_q^{l'}+\Phi))-f(-\pi+l(\alpha_p^l+\Phi),l'(\alpha_q^{l'}+\Phi)))\times\\ \times\frac{H_{\nu_p^l}^{(2)}(kR)\,e^{-li\nu_p^l\Omega}}{\frac{\partial}{\partial\nu}(H_{\nu}^{(2)}(ka^l)+Q^lH_{\nu}^{(2)}(ka^l))|_{\nu=\nu_p^l}}\,\frac{H_{\nu_q^{l'}}^{(2)}(kR')\,e^{-l'i\nu_q^{l'}\Omega'}}{\frac{\partial}{\partial\nu}(H_{\nu}^{(2)}(ka^{l'})+Q^{l'}H_{\nu}^{(2)}(ka^{l'}))|_{\nu=\nu_q^{l'}}} \quad (13)\end{aligned}$$

Let us notice that this expression has a form similar to the one defined by Thompson for the particular case of a discontinuity of material on a circular cylinder. Besides, we remark that, contrary of the expression of Albertsen et Christiansen, the operators $\mathcal{A}_{\alpha}^{\pm}$ are definite at first and second order.

We can then consider a development of f of second order without making diverge the expression. *It then becomes possible to consider, for example, the case of a curved plane with a discontinuity of material 'and' of curvature.*

3) Some validations concerning the expression of f for arbitrary wedge angle

We now compare the results obtained from (7) with the expressions known for some particular wedge angles, with $\Phi \geq \pi/2$, in the limit case $\sin\theta^{\pm} \to \infty$ so that $\Psi = 1$. Letting $\epsilon = \mathrm{sgn} Re\alpha'$, we choose to take $d^+ = (1+\epsilon)\Phi$ and $d^- = (1-\epsilon)\Phi$.

Denoting $a^\delta = a^+$ or a^- when $\delta = +1$ or -1, we can write

$$\chi(\alpha') = (\frac{1}{8\Phi}\,\partial_{\alpha'}\partial_{\varphi_\circ}\int_{-i\infty}^{i\infty} d\alpha\,[\mathrm{tg}(\frac{\pi}{2\Phi}(\alpha' - \epsilon\Phi - \alpha)\,\frac{\frac{\pi}{2\Phi}\sin\frac{\pi}{2\Phi}\alpha}{(\cos\frac{\pi}{2\Phi}\alpha + \epsilon\sin\frac{\pi}{2\Phi}\varphi_\circ)}$$
$$\times (\frac{1}{a^{-\epsilon}\sin\alpha} + \frac{1}{a^{\epsilon}\sin(\alpha + 2\epsilon\Phi)})]\,) + (\,i\,\frac{\epsilon}{a^\epsilon}\,\frac{\pi}{4\Phi}\,\partial_{\alpha'}\partial_{\varphi_\circ}[\frac{1}{\sin(\epsilon\Phi - \varphi_\circ)}$$
$$\times\,\mathrm{tg}(\frac{\pi}{4\Phi}(\alpha' + \varphi_\circ)) + \mathrm{tg}(\frac{\pi}{4\Phi}(\alpha' + \epsilon(\Phi - \pi)))\frac{\frac{\epsilon\pi}{2\Phi}\sin\frac{\pi}{2\Phi}\pi}{(\cos\frac{\pi}{2\Phi}\pi - \epsilon\sin\frac{\pi}{2\Phi}\varphi_\circ)}\,]\,), \qquad (14)$$

this expression being valid for $\alpha' \in]-3\Phi, 3\Phi[$, $\epsilon = \mathrm{sgn}\ Re\alpha'$.

In the case of a curved half-plane, $\Phi = \pi$ and $\epsilon a^\epsilon = a$, and so we obtain

$$\chi(\alpha') = (0) + i\,\frac{1}{4a}\,\partial_{\alpha'}\partial_{\varphi_\circ}[\frac{1}{\sin(\varphi_\circ)}\mathrm{tg}(\frac{1}{4}(\alpha' + \varphi_\circ)) - \frac{1}{2}\mathrm{tg}(\frac{1}{4}\alpha')\frac{1}{\sin(\frac{1}{2}\varphi_\circ)}]\,. \quad (15)$$

This gives us the second order term for far field function

$$(f_1(\pi + \varphi) - f_1(-\pi + \varphi))/k = \frac{i}{4ka}\,\partial_\varphi\partial_{\varphi_\circ}(\frac{\mathrm{tg}((\varphi + \varphi_\circ)/2)}{\cos(\varphi/2)\cos(\varphi_\circ/2)}), \quad (16)$$

which is conform to the result given in the paper of Borovikov [4], first developed by V.B. Filippov.

For the discontinuity of curvature, $\Phi = \frac{\pi}{2}$ and then the integral term for χ is

$$\frac{1}{4\pi}\,\partial_{\alpha'}\partial_{\varphi_\circ}\int_{-i\infty}^{i\infty} d\alpha\,[\frac{\cos\alpha'}{(\cos\alpha + \epsilon\sin\alpha')}\,\frac{1}{(\cos\alpha + \epsilon\sin\varphi_\circ)}(\frac{1}{a^+} - \frac{1}{a^-})] \qquad (17)$$
$$= \frac{i}{2\pi}\,\partial_{\alpha'}\partial_{\varphi_\circ}\,\frac{\epsilon\cos\alpha'}{(\sin\varphi_\circ - \sin\alpha')}(\,\frac{\pi/2 - \epsilon\alpha'}{\cos\alpha'} - \frac{\pi/2 - \epsilon\varphi_\circ}{\cos\varphi_\circ}\,)(\frac{1}{a^+} - \frac{1}{a^-})$$

for $-\pi/2 < \epsilon Re\alpha' < 3\pi/2$. The expression (14) for χ then gives us

$$(f_1(\pi + \varphi) - f_1(-\pi + \varphi))/k \;=\; -2i\frac{\cos\varphi\cos\varphi_\circ}{(\sin\varphi + \sin\varphi_\circ)^3}\,(\frac{1}{ka^+} - \frac{1}{ka^-}), \quad (15)$$

which is conform to the diffraction coefficient of Kaminetzky and Keller [3].

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