FOUR WAYS OF DEDUCING MAXWELL'S EQUATIONS FROM THEIR MICROSCOPIC COUNTERPARTS – LORENTZ'S THEORY OF ELECTRONS REVISITED

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ABSTRACT

A century ago, Lorentz deduced the macroscopic Maxwell equations by spatially averaging a set of postulated Maxwell equations for the microscopic electromagnetic field. This averaging procedure is still relevant for obtaining the macroscopic behaviour of a medium from the interaction between the microscopic field and the particles of the medium. Crucial in this process is the identification of the average of the microscopic quantities in terms of macroscopic quantities. In this paper it is investigated how the choice of the microscopic magnetic field quantity and the magnetic dipole model affects the identification of the average of the magnetic field quantity.

INTRODUCTION

In 1902, H.A. Lorentz published his article "The fundamental equations for electromagnetic phenomena in ponderable bodies deduced from the theory of electrons" [1]. In this article, Lorentz shows that the well-known Maxwell equations for the macroscopic electromagnetic field in matter may be deduced from postulated Maxwell equations for the microscopic electromagnetic field. The influence of the matter is represented by so-called 'electrons', small particles that give rise to a microscopic charge density and/or a microscopic electric current density. As such, these electrons cause the rapid spatial changes of the microscopic field. Different from the present-day terminology, Lorentz uses the word 'electrons' to indicate the entire family of (not necessarily negatively) charged particles, each member having a specific internal charge distribution and a specific motion. Depending on their properties, the electrons perform a certain role: there are conduction electrons, polarization electrons and magnetization electrons. The important step in his article is the spatial averaging of the unobservable microscopic quantities and the identification of their averages in terms of the observable macroscopic quantities. Lorentz shows that with the proper identifications, one indeed arrives at the macroscopic Maxwell equations.

The identification step mentioned above remains important for deducing the macroscopic behaviour of a medium from the interactions at the microscopic level, i.e. between the microscopic electromagnetic field and the particles of the medium. The identification of the spatial averages is not always trivial. This is illustrated by Lorentz who, in his article, considers a microscopic electric flux density, a microscopic magnetic field strength, and the electric current model (the Ampère model) for the magnetic dipoles. For this situation it turns out that the spatial average of the microscopic electric flux density is directly related to the macroscopic electric field strength, and the spatial average of the microscopic magnetic field strength is directly linked to the macroscopic magnetic flux density. The first question being addressed in the present paper is whether this somewhat odd relation between the microscopic and macroscopic field quantities may be avoided by choosing other quantities at the microscopic level. The second question is what the averaging and identification process yields for the alternative magnetic charge model of a magnetic dipole.

To answer these questions, in the next sections four different versions of microscopic equations will be used as a starting point. In all cases, the microscopic electric field strength will be used. The four cases are distinguished by considering either the magnetic flux density or the magnetic field strength, and either the electric current model or the magnetic charge model of a magnetic dipole. The conclusions are given at the end this paper. Without exception, SI units will be used.

THE ELECTRIC CURRENT MODEL

Microscopic magnetic flux density

First we will consider the case in which the microscopic electromagnetic field is represented by the microscopic electric field strength $\boldsymbol{e}$ and the microscopic magnetic flux density $\boldsymbol{b}$, and each magnetic dipole is represented by a small electrical current loop. For this situation we postulate the following Maxwell equations for the microscopic electromagnetic field

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{e} &= -\frac{\partial \boldsymbol{b}}{\partial t}, && (1) \\ \boldsymbol{\nabla} \times \boldsymbol{b} &= \mu_0 \boldsymbol{i} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{e}}{\partial t}, && (2) \end{aligned}$$

cf. [5]. Here $\boldsymbol{i}$ is the microscopic electric current density due to the motion of the various types of electrically charged particles inside the medium. Note that at the microscopic level the influence of the medium on the electromagnetic field only enters the equations through this current density, while the constitutive parameters are always those for the embedding of the particles, i.e. vacuum.

As a next step, the microscopic Maxwell equations are spatially averaged over a representative elementary domain. Such a domain is chosen large enough to contain so many particles that the behaviour of an individual particle has a negligible effect on the average of a quantity. On the other hand, the domain is taken small enough to let the average of a quantity follow all the changes that are observable at the macroscopic level. The averaging process turns (1) and (2) into

$$\begin{aligned} \boldsymbol{\nabla} \times \langle \boldsymbol{e} \rangle &= -\frac{\partial \langle \boldsymbol{b} \rangle}{\partial t}, && (3) \\ \boldsymbol{\nabla} \times \langle \boldsymbol{b} \rangle &= \mu_0 \langle \boldsymbol{i} \rangle + \mu_0 \varepsilon_0 \frac{\partial \langle \boldsymbol{e} \rangle}{\partial t}, && (4) \end{aligned}$$

where the brackets indicate that the spatial average of the enclosed microscopic quantity has been taken.

Before the identification of $\langle \boldsymbol{e} \rangle$ and $\langle \boldsymbol{b} \rangle$ with the macroscopic field quantities can take place, $\langle \boldsymbol{i} \rangle$ must be expressed in terms of macroscopic quantities. As explained by various authors [1 - 5], this average may be written as

$$\langle \boldsymbol{i} \rangle = \boldsymbol{I}_f + \boldsymbol{I}_p + \boldsymbol{I}_m. \qquad (5)$$

This is the total electric current density due to the motion of both the free and the bound electric charges. Each term represents the contribution of a specific group of particles. The term $\boldsymbol{I}_f$ is due to the motion of free, electrically charged particles, and equals the current density $\boldsymbol{J}$. The term $\boldsymbol{I}_p$ is caused by the motion of the bound electric charges of the electric dipoles. It corresponds to the polarization current density and depends on the polarization $\boldsymbol{P}$ of the medium

$$\boldsymbol{I}_p = \frac{\partial \boldsymbol{P}}{\partial t}. \qquad (6)$$

The term $\boldsymbol{I}_m$ is due to the motion of the bound electric charges of the magnetic dipoles. This term is equal to the magnetization current density and depends on the magnetization $\boldsymbol{M}$ of the medium

$$\boldsymbol{I}_m = \boldsymbol{\nabla} \times \boldsymbol{M}. \qquad (7)$$

Substitution of (5) - (7) into (3) and (4) yields

$$\begin{aligned} \boldsymbol{\nabla} \times \langle \boldsymbol{e} \rangle &= -\frac{\partial \langle \boldsymbol{b} \rangle}{\partial t}, && (8) \\ \mu_0^{-1} \boldsymbol{\nabla} \times \langle \boldsymbol{b} \rangle - \boldsymbol{\nabla} \times \boldsymbol{M} &= \boldsymbol{J} + \varepsilon_0 \frac{\partial \langle \boldsymbol{e} \rangle}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t}, && (9) \end{aligned}$$

where we have applied a rearrangement of some terms in the second equation.

Now we may identify $\langle \boldsymbol{e} \rangle$ and $\langle \boldsymbol{b} \rangle$ in terms of the macroscopic field quantities. To achieve this, we consider the Maxwell equations for the macroscopic field quantities $\{\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}\}$. After substitution of the relations

$$\begin{aligned} \boldsymbol{D} &= \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}, && (10) \\ \boldsymbol{B} &= \mu_0 (\boldsymbol{H} + \boldsymbol{M}), && (11) \end{aligned}$$

in the second of these Maxwell equations, the following macroscopic equations are obtained

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, && (12) \\ \mu_0^{-1} \boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{\nabla} \times \boldsymbol{M} &= \boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t}. && (13) \end{aligned}$$

When we compare these equations to (8) and (9), it becomes obvious that we must make the identifications

$$\begin{aligned} \langle \boldsymbol{e} \rangle &= \boldsymbol{E}, && (14)\\ \langle \boldsymbol{b} \rangle &= \boldsymbol{B}. && (15) \end{aligned}$$

These identifications also appear in [5]. Provided we assume conservation of charge at the microscopic and the macroscopic level, we may show that such identifications are consistent with the local electric and magnetic Gauss laws.

Microscopic magnetic field strength

When we repeat the above analysis for the case in which the microscopic magnetic field is represented by the microscopic magnetic field strength, we must make the identifications

$$\begin{aligned} \langle \boldsymbol{e} \rangle &= \boldsymbol{E}, && (16)\\ \langle \boldsymbol{h} \rangle &= \mu_0^{-1}\boldsymbol{B}. && (17) \end{aligned}$$

The latter identification may also be found in [1, 2].

THE MAGNETIC CHARGE MODEL

Microscopic magnetic field strength

Here we will investigate the situation in which the microscopic electromagnetic field is represented by the microscopic electric field strength $\boldsymbol{e}$ and the microscopic magnetic field strength $\boldsymbol{h}$, and each magnetic dipole is represented by two opposing magnetic charges that are separated by a small distance. For this case we postulate the microscopic Maxwell equations

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{e} &= -\boldsymbol{i}^* - \mu_0 \frac{\partial \boldsymbol{h}}{\partial t}, && (18)\\ \boldsymbol{\nabla} \times \boldsymbol{h} &= \boldsymbol{i} + \varepsilon_0 \frac{\partial \boldsymbol{e}}{\partial t}. && (19) \end{aligned}$$

In addition to the microscopic electric current density $\boldsymbol{i}$, there now occurs the microscopic magnetic current density $\boldsymbol{i}^*$ due to the motion of the magnetically charged particles inside the medium.

Next, the microscopic Maxwell equations are spatially averaged over a representative elementary domain. This turns (18) and (19) into

$$\begin{aligned} \boldsymbol{\nabla} \times \langle \boldsymbol{e} \rangle &= -\langle \boldsymbol{i}^* \rangle - \mu_0 \frac{\partial \langle \boldsymbol{h} \rangle}{\partial t}, && (20)\\ \boldsymbol{\nabla} \times \langle \boldsymbol{h} \rangle &= \langle \boldsymbol{i} \rangle + \varepsilon_0 \frac{\partial \langle \boldsymbol{e} \rangle}{\partial t}, && (21) \end{aligned}$$

in agreement with the analysis given in [2].

Before the identification of $\langle \boldsymbol{e} \rangle$ and $\langle \boldsymbol{b} \rangle$ can take place, $\langle \boldsymbol{i} \rangle$ and $\langle \boldsymbol{i}^* \rangle$ must be expressed in terms of macroscopic quantities. The first average may now be written as [2]

$$\langle \boldsymbol{i} \rangle = \boldsymbol{I}_f + \boldsymbol{I}_p, \tag{22}$$

where $\boldsymbol{I}_f$ is the electric current density $\boldsymbol{J}$ and $\boldsymbol{I}_p$ is the polarization current density as given in (6). The second average turns out to be [2]

$$\langle \boldsymbol{i}^* \rangle = \boldsymbol{I}_m^*. \tag{23}$$

Here, $\boldsymbol{I}_m^*$ is caused by the motion of the bound magnetic charges of the magnetic dipoles. It is equal to the magnetic polarization current density and depends on the polarization $\boldsymbol{M}$ of the medium

$$\boldsymbol{I}_m^* = \mu_0 \frac{\partial \boldsymbol{M}}{\partial t}. \tag{24}$$

We see that the contribution due to the magnetic dipoles has disappeared from (22) and now shows up in (23). Substitution of (22) - (24) into (20) and (21) yields

$$\nabla \times \langle \boldsymbol{e} \rangle = -\mu_0 \frac{\partial \langle \boldsymbol{h} \rangle}{\partial t} - \mu_0 \frac{\partial \boldsymbol{M}}{\partial t}, \tag{25}$$

$$\nabla \times \langle \boldsymbol{h} \rangle = \boldsymbol{J} + \varepsilon_0 \frac{\partial \langle \boldsymbol{e} \rangle}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t}, \tag{26}$$

where we have applied a rearrangement of some terms in both equations.

Now we may identify $\langle \boldsymbol{e} \rangle$ and $\langle \boldsymbol{h} \rangle$ in terms of the macroscopic field quantities. Substitution of the relations

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}, \tag{27}$$

$$\boldsymbol{B} = \mu_0 (\boldsymbol{H} + \boldsymbol{M}), \tag{28}$$

in the Maxwell equation for the macroscopic field quantities $\{\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}\}$ gives

$$\nabla \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} - \mu_0 \frac{\partial \boldsymbol{M}}{\partial t}, \tag{29}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{P}}{\partial t}. \tag{30}$$

When we compare these equations to (25) and (26) we arrive at the identifications

$$\langle \boldsymbol{e} \rangle = \boldsymbol{E}, \tag{31}$$

$$\langle \boldsymbol{h} \rangle = \boldsymbol{H}. \tag{32}$$

These identifications also appear in [2].

Microscopic magnetic flux density

When we repeat the above analysis for the case in which the microscopic magnetic field is represented by the microscopic magnetic flux density, we arrive at the identifications

$$\langle \boldsymbol{e} \rangle = \boldsymbol{E}, \tag{33}$$

$$\langle \boldsymbol{b} \rangle = \mu_0 \boldsymbol{H}. \tag{34}$$

CONCLUSIONS

Following Lorentz, we have applied a spatial averaging of several versions of microscopic Maxwell equations, and have identified the averages of the corresponding microscopic field quantities in terms of the macroscopic field quantities. For the magnetic field it has turned out that the identifications depend on the microscopic magnetic field quantity and the magnetic dipole model under consideration. Based on the results in (14) - (17) and (31) - (34), we think that it is most natural to apply the microscopic magnetic flux density in case of the electrical current model and the microscopic magnetic field strength in case of the magnetic charge model, since the averages of the microscopic field quantities are then equal to the corresponding macroscopic field quantities. From (17) and (32) it follows that the averages of the microscopic magnetic field strength for the electric current model and the magnetic charge model differ by $\mu_0^{-1}\boldsymbol{B} - \boldsymbol{H} = \boldsymbol{M}$. The explanation of this fact is that the microscopic magnetic field strength near a dipole is different for both models [2].

References

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