ANALYSIS OF ELECTROMAGNETIC DIFFRACTION BY WEDGES WITH THE METHOD OF EDGE FUNCTIONS

Andrey Osipov

German Aerospace Center, Institute of Radio Frequency Technology and Radar Systems, Oberpfaffenhofen, 82234 Wessling, Germany
E-mail: Andrey.Osipov@dlr.de

Abstract

Edge functions are local solutions of Maxwell's equations in wedge-shaped domains, which are not required to meet radiation conditions. The paper shows how these functions can be applied to analysis of a canonical problem of electromagnetic diffraction by a dielectric wedge.

Introduction

Presence of edges in a scattering surface may considerably influence behavior of electromagnetic fields interacting with the surface. This influence manifests itself with field singularities at the edges and specific edge-diffracted contributions in the far field [1]. Simulation of those important effects is complicated by the difficulties with obtaining analytical solutions of Maxwell's equations for wedges with imperfectly reflecting boundaries. The goal of this paper is to demonstrate that a newly established method of edge functions (MEF, [2]) has a great potential for accurate analysis of electromagnetic fields in a broad variety of practically interesting wedge-shaped geometries. The method can be seen as a modification of an approach introduced by Bates [3] and employed later to study two-dimensional diffraction of an E-polarized electromagnetic wave by a dielectric wedge [4]. In contrast to the Bates' approach, MEF includes the fractional-order Bessel functions and is not limited to a specific polarization.

Application of MEF to the analysis of electromagnetic diffraction and scattering by perfectly conducting and impedance wedges has been discussed in [2]. This paper extends the approach to the case of a dielectric wedge shown in Fig.1. Our focus will be on description of a computationally efficient algorithm for generating the edge functions (EFs) of the configuration. We shall also discuss the combining of EFs into full-wave solutions consistent with radiation conditions. Finally we show how the MEF solutions can be used to extract information about the low- and high-frequency behaviour of fields diffracted by dielectric wedges. Time dependence $\exp(-i\omega t)$ is assumed and omitted throughout the paper.

Generating Edge Functions

Consider a wedge with dielectric permittivity $\epsilon = \epsilon_2$ and magnetic permeability $\mu = \mu_2$ immersed in a medium with $\epsilon = \epsilon_1$ and $\mu = \mu_1$ (Fig. 1). In the cylindrical coordinate system associated with the edge, the wedge faces are defined by the equations: $0 \le \rho < \infty$, $-\infty < z < \infty$, and $\phi = \pm\Phi$ ($\pi/2 \le \Phi \le \pi$). Without loss of generality we may assume that the electromagnetic field depends on the z coordinate as $\exp(ik_z z)$ with k_z being a constant, which allows us to express all the field components through z components of the electric and magnetic fields, E_z and H_z, respectively. Next we factor out and omit the z-dependence from the field equations so as to formulate the diffraction problem in terms of two scalar amplitudes $\tilde{E}_z$ and $\tilde{H}_z$, where the tilde indicates functions of ρ and ϕ only. Furthermore, by using the symmetry of the configuration with respect to the bisection plane $\phi = 0$ and introducing symmetric and antisymmetric fields, $\tilde{E}_z^{(s,a)} = \left[\tilde{E}_z(\rho,\phi) \pm \tilde{E}_z(\rho,-\phi)\right]/2$ and $\tilde{H}_z^{(s,a)} = \left[\tilde{H}_z(\rho,\phi) \pm \tilde{H}_z(\rho,-\phi)\right]/2$, we split the problem into two uncoupled subproblems for $(\tilde{E}_z^a, \tilde{H}_z^s)$ and $(\tilde{E}_z^s, \tilde{H}_z^a)$ defined in half-space $0 \le \phi \le \pi$. Because of the duality of Maxwell's equations, it is sufficient to solve one of these subproblems, say that for $\tilde{E}_z^a$ and $\tilde{H}_z^s$, since solution of another one is obtained by the substitutions: $\epsilon \to \mu$, $\mu \to \epsilon$, $\tilde{E}_z^a \to \tilde{H}_z^a$, $\tilde{H}_z^s \to -\tilde{E}_z^s$.

Thus, the problem to solve consists in finding a pair of functions $\tilde{E}_z^a$ and $\tilde{H}_z^s$ that are solutions of scalar Helmholtz equations, meeting the edge conditions at $\rho = 0$ and ensuring continuity of tangential field components

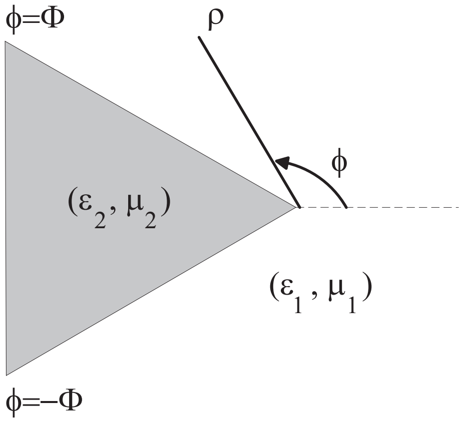

Figure 1: The dielectric wedge geometry

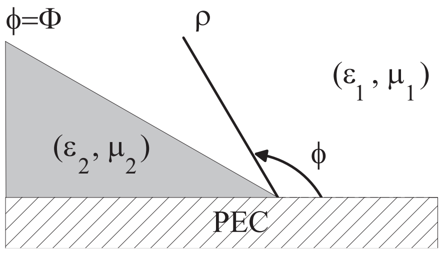

Figure 2: The half-wedge configuration

$\tilde{E}^a_\rho$, $\tilde{E}^a_z$, $\tilde{H}^s_\rho$, and $\tilde{H}^s_z$ across the dielectric interface $\phi = \Phi$. The boundary conditions at $\phi = 0, \pi$ correspond to the presence of a perfectly conducting wall (Fig. 2). Note that the radiation conditions at $\rho \to \infty$ are not imposed because we look for local solutions of Maxwell's equation, valid at the edge and in its finite vicinity.

Let us search for the solution in the form of the following Bessel function expansions:

$$\left\{ \begin{array}{c} \tilde{E}^a_z \\ \\ Z_1 \tilde{H}^s_z \end{array} \right\} = \sum_{m=0}^{\infty} J_{\tau+2m}\left(\rho\sqrt{k_1^2 - k_z^2}\right) \left\{ \begin{array}{c} A^{(1)}_m \sin\left[\phi\left(\tau + 2m\right)\right] \\ \\ B^{(1)}_m \cos\left[\phi\left(\tau + 2m\right)\right] \end{array} \right\} \qquad \text{for} \quad 0 \le \phi \le \Phi, \tag{1}$$

$$\left\{ \begin{array}{c} \tilde{E}^a_z \\ \\ Z_2 \tilde{H}^s_z \end{array} \right\} = \sum_{m=0}^{\infty} J_{\tau+2m}\left(\rho\sqrt{k_2^2 - k_z^2}\right) \left\{ \begin{array}{c} A^{(2)}_m \sin\left[\left(\phi - \pi\right)\left(\tau + 2m\right)\right] \\ \\ B^{(2)}_m \cos\left[\left(\phi - \pi\right)\left(\tau + 2m\right)\right] \end{array} \right\} \qquad \text{for} \quad \Phi \le \phi \le \pi, \tag{2}$$

where $Z = \sqrt{\mu/\epsilon}$ and $k = \omega\sqrt{\mu\epsilon}$ are the intrinsic impedance and the wave number of the media, whereas parameter τ is to be specified later. Representations (1) and (2) are clearly solutions of the Helhmoltz equations satisfied by $\tilde{E}^a_z$ and $\tilde{H}^s_z$. The choice of trigonometric functions in (1) and (2) corresponds to the boundary conditions at $\phi = 0, \pi$. By choosing τ such that Re $\tau \ge 0$ we obtain fields compliant with the edge conditions.

Expansion coefficients in (1) and (2) are derived by enforcing continuity of the tangential fields components at the wedge face $\phi = \Phi$ as $\rho \to 0$. Expanding the Bessel functions in power series with respect to ρ and equating terms with equal powers of ρ leads to a system of recurrent relations between the expansion coefficients,

$$\mathbf{L}_m \vec{X}_m = \vec{g}_m\left(\vec{X}_0, \vec{X}_1, \ldots, \vec{X}_{m-1}\right), \qquad m = 0, 1, 2, \ldots \tag{3}$$

where $\vec{X}_m = \left\{A^{(1)}_m, B^{(1)}_m, A^{(2)}_m, B^{(2)}_m\right\}'$ is a column vector and $\mathbf{L}_m$ is a matrix given by the formula

$$\mathbf{L}_m = \begin{pmatrix} S^{(1)}_m & 0 & -\eta_{21}^{\tau+2m} S^{(2)}_m & 0 \\ 0 & C^{(1)}_m & 0 & -\eta_{21}^{\tau+2m} Z_{12} C^{(2)}_m \\ k_{z1} S^{(1)}_m & -S^{(1)}_m & -k_{z1}\eta_{21}^{\tau+2m-2} S^{(2)}_m & k_{21}\eta_{21}^{\tau+2m-2} S^{(2)}_m \\ -C^{(1)}_m & k_{z1} C^{(1)}_m & k_{21}\eta_{21}^{\tau+2m-2} Z_{12} C^{(2)}_m & -k_{z1}\eta_{21}^{\tau+2m-2} Z_{12} C^{(2)}_m \end{pmatrix}. \tag{4}$$

Here $Z_{12} = Z_1/Z_2$, $k_{21} = k_2/k_1$, $k_{z1} = k_z/k_1$, $\eta_{21} = \sqrt{\left(k_{21}^2 - k_{z1}^2\right)/\left(1 - k_{z1}^2\right)}$, $S^{(1)}_m = \sin\left[\left(\tau + 2m\right)\Phi\right]$, $C^{(1)}_m = \cos\left[\left(\tau + 2m\right)\Phi\right]$, $S^{(2)}_m = \sin\left[\left(\tau + 2m\right)\left(\Phi - \pi\right)\right]$, and $C^{(2)}_m = \cos\left[\left(\tau + 2m\right)\left(\Phi - \pi\right)\right]$. Column vector $\vec{g}_m = \left\{g_1^{(m)}, g_2^{(m)}, g_3^{(m)}, g_4^{(m)}\right\}'$ is expressed in terms of elements of $\vec{X}_j$ with $j = 0, 1, \ldots, m-1$ as follows:

$$g_1^{(m)} = -\sum_{q=0}^{m-1} \frac{(-1)^{m-q}}{(m-q)!}\chi_{mq}\left[A^{(1)}_q S^{(1)}_q - \eta_{21}^{\tau+2m} A^{(2)}_q S^{(2)}_q\right], \tag{5}$$

$$g_2^{(m)} = -\sum_{q=0}^{m-1} \frac{(-1)^{m-q}}{(m-q)!}\chi_{mq}\left[B^{(1)}_q C^{(1)}_q - \eta_{21}^{\tau+2m} Z_{12} B^{(2)}_q C^{(2)}_q\right],$$

$$g_3^{(m)} = -\sum_{q=0}^{m-1} \tfrac{(-1)^{m-q}}{(m-q)!}\chi_{mq} \left[\left(k_{z1}A_q^{(1)} - \tfrac{\tau+2q}{\tau+2m}B_q^{(1)}\right)S_q^{(1)} - \left(k_{z1}A_q^{(2)} - k_{21}\tfrac{\tau+2q}{\tau+2m}B_q^{(2)}\right)\eta_{21}^{\tau+2m-2}S_q^{(2)}\right],$$
$$g_4^{(m)} = -\sum_{q=0}^{m-1} \tfrac{(-1)^{m-q}}{(m-q)!}\chi_{mq} \left[\left(k_{z1}B_q^{(1)} - \tfrac{\tau+2q}{\tau+2m}A_q^{(1)}\right)C_q^{(1)} - \left(k_{z1}B_q^{(2)} - k_{21}\tfrac{\tau+2q}{\tau+2m}A_q^{(2)}\right)\eta_{21}^{\tau+2m-2}Z_{12}C_q^{(2)}\right],$$

where $\chi_{mq} = (\tau+2m)(\tau+2m-1)\dots(\tau+m+q+1)$ and $m \geq 1$. If $m=0$, $\vec{g}_0 = \{0,0,0,0\}'$.

Solution of (3) begins with the lowest order equation ($m=0$) which is a source-free system of 4 linear algebraic equations with respect to $A_0^{(1)}$, $B_0^{(1)}$, $A_0^{(2)}$, and $B_0^{(2)}$. A non-trivial solution exists if the determinant of the system matrix vanishes, $\det \mathbf{L}_0 = 0$. Since

$$\det \mathbf{L}_m = k_{21}\eta_{21}^{2(\tau+2m-1)}\Lambda_h(\tau+2m)\,\Lambda_e(\tau+2m), \tag{6}$$

with

$$\Lambda_h(\tau) = \cos(\tau\Phi)\sin[\tau(\Phi-\pi)] - \tfrac{\mu_1}{\mu_2}\sin(\tau\Phi)\cos[\tau(\Phi-\pi)], \tag{7}$$
$$\Lambda_e(\tau) = \cos(\tau\Phi)\sin[\tau(\Phi-\pi)] - \tfrac{\epsilon_2}{\epsilon_1}\sin(\tau\Phi)\cos[\tau(\Phi-\pi)], \tag{8}$$

the solvability condition is that τ must be a zero of either (7) or (8). Each of the latter functions has infinite number of roots (τ_n^h and τ_n^e, respectively) that can be arranged according to the value of their real parts such that Re $\tau_n^{h,e} > 0$ for $n = 1, 2, \dots, +\infty$, Re $\tau_n^{h,e} < 0$ for $n = -1, -2, \dots, -\infty$, and $\tau_n^{h,e} = 0$ for $n = 0$. The zeros with $n < 0$ are not allowed by the edge conditions, implying that parameter τ in (1) and (2) is specified by the relations: $\tau = \tau_n^{h,e}$ with $n = 0, 1, 2, \dots$

With a properly chosen value of τ the zero order system in (3) is shown to have non-trivial solutions of the form $\vec{X}_0 = A_0^{(1)}\{1, \alpha, \beta, \gamma\}'$ with

$$\alpha \equiv \frac{B_0^{(1)}}{A_0^{(1)}} = \frac{Z_{12}k_{z1}\left(\eta_{21}^2 - 1\right)}{\eta_{21}^2 Z_{12} - k_{21}\cot(\tau\Phi)\tan[\tau(\Phi-\pi)]}, \tag{9}$$
$$\beta \equiv \frac{A_0^{(2)}}{A_0^{(1)}} = \frac{\eta_{21}^{-\tau}\sin(\tau\Phi)}{\sin[\tau(\Phi-\pi)]}, \quad \gamma \equiv \frac{B_0^{(2)}}{A_0^{(1)}} = \alpha\frac{\eta_{21}^{-\tau}\cos(\tau\Phi)}{Z_{12}\cos[\tau(\Phi-\pi)]}.$$

Coefficient $A_0^{(1)}$ is arbitrary and plays the role of a normalization factor.

Once $\vec{X}_0$ is specified, source term $\vec{g}_1$ can be determined from (5) and inserted into (3) with $m = 1$ to give algebraic equations for specification of $\vec{X}_1$. In contrast to the case $m = 0$, this system is inhomogeneous. Matrix $\mathbf{L}_1$ of the system is a 4 by 4 matrix that can be analytically inverted, given that $\tau + 2$ is not a zero of functions (7) or (8). If the latter condition is true, one obtains a unique and bounded solution $\vec{X}_1$. By repeating the procedure described above, and assuming that for all natural values of m numbers $\tau + 2m$ are not zeros of (7) or (8), one constructs the whole sequence of vectors $\vec{X}_m$ and, thus, completely defines expansions (1) and (2).

The recurrent construction of the expansion coefficients is conditioned by the made assumption about distribution of zeros of (7) and (8). It can be shown that the applicability condition can be formulated entirely in terms of the wedge angle Φ. Let τ be a zero of (7) or (8) with

$$\Phi = \frac{\pi}{2}\left(1 + \frac{l}{j}\right), \tag{10}$$

where l and j are mutually simple integer numbers such that $j \geq 1$ and $-j < l \leq j$. Then $\tau + jm$ with $m = 2, 4, 6, \dots$ is also a zero of the same equation. If both l and j are odd, then m can be an arbitrary integer. The problem with the recurrent construction of solutions for such "simple" wedge angles is analogous to a similar problem arising in the theory of Meixner's series [5, 6, 7] and can be avoided by slightly changing the value of Φ so as to make Φ/π an irrational number.

Given that (10) is not true, one may safely employ recurrent procedure (3) for each $\tau = \tau_n^{h,e}$ with $n = 0, 1, 2, \dots$ to generate a set of solutions of the form (1) and (2),

$$\vec{\Psi}_n^{h,e}(\rho,\phi) \equiv \left\{\tilde{E}_z^a,\ Z\tilde{H}_z^s\right\}'_{\tau=\tau_n^{h,e}} \tag{11}$$

which are designated as EFs. It is important to notice that EFs can not be regarded as the true eigenfunctions of the boundary value problem because they do not meet one of the boundary conditions. Similarly, the set of numbers $\tau_n^{h,e}$ is not the spectrum in the classical sense of this term.

Applying Edge Functions

In order to account for the radiation conditions and construct a global solution of the diffraction problem, one may consider a superposition of EFs,

$$\left\{ \begin{array}{c} \tilde{E}_z^a \\ \\ Z\tilde{H}_z^s \end{array} \right\} = \sum_{n=0}^{\infty} C_n^h \vec{\Psi}_n^h + \sum_{n=0}^{\infty} C_n^e \vec{\Psi}_n^e \tag{12}$$

with coefficients to be adjusted so as to make (12) compliant with the conditions at $\rho = \infty$. Assume for example that the incident field is a plane wave. Then the radiation conditions can be written in terms of a difference between the exact and geometrical optics fields, as follows

$$\left(\frac{\partial}{\partial \rho} - ik \right) \left(\left\{ \begin{array}{c} \tilde{E}_z^a \\ \\ Z\tilde{H}_z^s \end{array} \right\} - \left\{ \begin{array}{c} \tilde{E}_z^a \\ \\ Z\tilde{H}_z^s \end{array} \right\}_{GO} \right) = o\left(\rho^{-\frac{3}{2}}\right) \tag{13}$$

where $0 < \phi < \pi$ and $\rho \to \infty$. The geometrical optics field is derived according to the laws of geometrical optics and is, therefore, easily available. By setting $\rho = R$ with R chosen such that the right-hand side in (13) can be neglected, we obtain functional equations to be satisfied by a proper choice of $C_n^{h,e}$ with $n = 0, 1, \ldots$. The simplest way to determine the coefficients is to reduce the functional equations to a system of linear algebraic equations, which can be achieved, for example, by collocating at a number of sampling points distributed on the circle $\rho = R$ and $0 < \phi < \pi$.

By definition, EFs are given by linear combinations of Bessel functions, which makes them particularly well suited for the analysis of the field behavior near edges. When $\rho \to 0$, the Bessel functions can be replaced with their small-argument expansions, which leads to field representations in the form of power series – the so-called Meixner's series. As it is obvious from (1) and (2), EFs represent such solutions of Maxwell's equations, whose z-components behave as $O(\rho^\tau)$ with τ defined by certain solvability conditions. In the case of a dielectric wedge these conditions are given by (7) and (8) together with a pair of dual equations resulting from the first two on the substitutions: $\epsilon \to \mu$, $\mu \to \epsilon$. One can easily check that these four equations are equivalent to the equations derived by Meixner in [5], implying that MEF recovers the exact order of the field singularity at the edge predicted by Meixner's asymptotic method. An important advantage of MEF over Meixner's method is the ease of obtaining higher-order terms, which in the framework of MEF is achieved by recurrently solving systems of linear algebraic equations, whereas Meixner's approach is based on recurrent systems of ordinary differential equations. One further advantage is the MEF capability to provide global solutions of diffraction problems, relating thereby the field behaviour at the edge to a given excitation at infinity.

MEF solutions cannot be directly applied to the far-field analysis because of the slowing convergence of the EF series as the distance from the edge increases. This problem can be overcome by employing uniform high-frequency asymptotic representations of the electromagnetic field diffracted by a wedge, which allows one to express the diffraction coefficient through the total field. This permits accurate determination of the diffraction coefficient based on the knowledge of the total field at a finite distance from the edge [2].

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