# A RIGOROUS APPROACH TO WAVE SCATTERING BY ELLIPTIC DISKS AND RELATED PROBLEMS

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## ABSTRACT

An original approach to solve rigorously a class of wave scattering problems from thin elliptic disks is developed. Applying the boundary conditions, this non-traditional formulation first establishes two-dimensional dual integral equations with trigonometric function kernels for an unknown spectral density function. A parametrisation technique transforms these equations to "disk-like" one-dimensional dual integral equations with Bessel function kernels, that are solved by the Method of Regularisation. Finally, each problem is reduced to the solution of an infinite system of linear equations of the second kind for unknown coefficients arising from representation of the unknown spectral density function by a Neumann's series.

## INTRODUCTION

The thin plate of finite extent with highest symmetry is a circular disk. Various wave scattering problems for circular disks have been comprehensively examined by many authors. Breaking this symmetry introduces a complexity which seems to be unsolvable by rigorous methods; mostly these problems are solved by purely numerical techniques. The main drawback of these techniques is well-known: it is impossible to make a valid estimate of the accuracy of calculations and to guarantee the accuracy of the estimated physical quantities. This limitation is common to all purely numerical techniques applied to first-kind Fredholm equations. Certainly, second-kind Fredholm equations are preferable since the Fredholm alternative guarantees the convergence of the solution to the finitely truncated system to the true solution. The main problem is how to obtain such well-behaved equations. To solve this problem we use the Method of Regularisation (MoR) to transform the original functional equations, corresponding a first-kind Fredholm equation, to a well-conditioned second-kind Fredholm equation in matrix formulation.

The key point of the MoR is the splitting of the initial problem operator into singular (or, with some care, "static" part) and regular (smooth) parts and the analytical construction of the inverse of the singular operator. The next step is the application of the inverse operator to the split operator, yielding the sum of the identity operator and a completely-continuous operator. Thus the posed problem is finally reduced to the desired format. Many practical examples of this approach may be found in [1]. In this paper we extend the MoR to solve a new class of wave scattering problems for "disk-like" structures shaped as elliptic plates. In contrast to the analogous problem for circular disks, this problem has been largely untouched by rigorous approaches: the most significant results were achieved in [2],[3]. In this paper we restrict ourselves to one problem: plane wave scattering by acoustically hard elliptic plate, but note that the same approach extends (with some additional complexity) to various other acoustic and EM wave-scattering problems from these elliptic plates. The case of electromagnetic plane wave incidence is characterised by the mutual coupling of the TE- and TM-modes resulting in a *coupled* infinite system of linear algebraic equations (i.s.l.a.e.).

## PROBLEM STATEMENT

Let the acoustically hard elliptic plate with minor and major semi-axes $a$ and $b$, respectively, lie in the plane $z = 0$ with centre located at the origin (see Figure 1); let $a/b < 1$. It is excited by the normally incident plane wave with velocity potential $U_0 = \exp(ikz)$ where the harmonic time dependence $\exp(-i\omega t)$ is supressed throughout the text. Decompose the total field as the sum $U^{tot} = U_0 + U^{sc}$, where the scattered field in

Cartesian coordinates $(x, y, z)$ is representable in the form

$$U^{sc}(x,y,z) = sgn(z)\int_0^\infty d\nu \cos\nu x \int_0^\infty d\mu g(\nu,\mu)\cos\mu y e^{-|z|\sqrt{\nu^2+\mu^2-k^2}}. \tag{1}$$

The unknown spectral density $g$ is to be determined. The Sommerfeld radiation conditions require the choice $Im(\sqrt{\nu^2+\mu^2-k^2}) < 0$. The boundary conditions are

$$\left.\frac{\partial U^{tot}}{\partial z}\right|_{z=+0} = \left.\frac{\partial U^{tot}}{\partial z}\right|_{z=-0} = 0, \quad \sqrt{\frac{x^2}{b^2}+\frac{y^2}{a^2}} < 1, \tag{2}$$

$$U^{sc}(x,y,+0) = U^{sc}(x,y,-0), \quad \sqrt{\frac{x^2}{b^2}+\frac{y^2}{a^2}} > 1. \tag{3}$$

**PROBLEM SOLUTION**

Enforcement of the boundary conditions leads to the two-dimensional dual integral equations for the unknown function $G(\nu,\tau) = g(\nu, \sqrt{\tau^2-\nu^2}/q)$, where $q = a/b$:

$$\int_0^\infty d\tau\tau J_0(\tau b\rho)\int_0^\tau d\nu G(\nu,\tau)\sqrt{\frac{\tau^2-(1-q^2)\nu^2-q^2k^2}{\tau^2-\nu^2}} = ikq^2, \quad \rho < 1, \tag{4}$$

$$\int_0^\infty d\tau\tau J_0(\tau b\rho)\int_0^\tau d\nu \frac{G(\nu,\tau)}{\sqrt{\tau^2-\nu^2}} = 0, \quad \rho > 1. \tag{5}$$

To obtain these equations we used the natural parametrisation $x = b\rho\cos\varphi$, $y = a\rho\sin\varphi$ and expansion of the product of the cosine functions in a series involving Chebyshev polynomials (see [1], Part I, p.297). The next step is the representation of $G$ in a Neumann series

$$G(\nu,\tau) = \frac{b^{1/2}}{(2\pi)^{1/2}\tau^{3/2}}\sum_{k=0}^\infty y_k J_{2k+3/2}(\tau b) \tag{6}$$

where the coefficients $y_k$ are to be determined. This representation automatically satisfies the equation (5) and transforms equation (4) into the equivalent form

$$\pi^{-1/2}b^{-1}(1-\rho^2)^{-1/2}\sum_{k=0}^\infty \frac{\Gamma(k+3/2)}{\Gamma(k+1)} y_k P_{2k+1}(\sqrt{1-\rho^2}) =$$

$$ik\frac{q^2}{E(\sqrt{1-q^2})} + \frac{b^{1/2}}{(2\pi)^{1/2}}\sum_{k=0}^\infty y_k \int_0^\infty d\tau\tau^{1/2}J_{2k+3/2}(\tau b)Q(\tau;q,k)J_0(\tau b\rho), \quad \rho < 1 \tag{7}$$

where

$$Q(\tau;q,k) = 1 - \frac{1}{\tau E(\sqrt{1-q^2})}\int_0^\tau \sqrt{\frac{\tau^2-(1-q^2)\nu^2-q^2k^2}{\tau^2-\nu^2}}d\nu. \tag{8}$$

$Q(\tau;q,k)$ is an asymptotically small parameter that obeys $Q(\tau;q,k) = O(k^2/\tau^2)$ as $\tau \to \infty$. Use the orthogonality of the functions $P_{2k+1}(\sqrt{1-\rho^2})$ with respect to the weight function $\rho(1-\rho^2)^{-1/2}$ on $(0,1)$ to obtain the following second kind i.s.l.a.e. for the rescaled coefficients $Y_n = (4n+3)^{-1/2}y_n$,

$$Y_n - \sum_{k=0}^\infty Y_k S_{kn}^{(R)} = W_n^{(R)}, \tag{9}$$

where $n = 0, 1, 2, ...,$ and

$$S_{kn}^{(R)} = [(4n+3)(4k+3)]^{1/2} \int_0^\infty \tau^{-1} Q(\tau; q, k) J_{2k+3/2}(\tau b) J_{2n+3/2}(\tau b) d\tau, \tag{10}$$

$$W_n^{(R)} = \frac{2i}{\sqrt{3}} \frac{q^2 kb}{E(\sqrt{1-q^2})} \delta_{0n}. \tag{11}$$

The integral

$$R^{(R)}(\tau; q, k) = \tau^{-1} \int_0^\tau \sqrt{\frac{\tau^2 - (1-q^2)\nu^2 - q^2k^2}{\tau^2 - \nu^2}} d\nu \tag{12}$$

contained in (8) may be evaluated in terms of complete elliptic integrals of the first and second kind, depending upon relationship between $\tau$ and the wavenumbers $qk$ and $k$. For example, when $\tau < qk$,

$$R^{(R)}(\tau; q, k) = -i\sqrt{1-q^2} \int_0^1 \sqrt{\frac{b^2 + x^2}{1 - x^2}} dx = -iq \frac{\sqrt{k^2 - \tau^2}}{\tau} E\left( \frac{\sqrt{1-q^2}\tau}{q\sqrt{k^2 - \tau^2}} \right) \tag{13}$$

where $b = (q^2k^2 - \tau^2)/(1-q^2)\tau^2$; and when $\tau > k$,

$$R^{(R)}(\tau; q, k) = \sqrt{1-q^2} \int_0^1 \sqrt{\frac{a^2 - x^2}{1 - x^2}} dx = \frac{\sqrt{\tau^2 - q^2k^2}}{\tau} E\left( \frac{\sqrt{1-q^2}\tau}{\sqrt{\tau^2 - q^2k^2}} \right). \tag{14}$$

The value of this integral when $qk < \tau < k$ may be found in [1].

## ILLUSTRATIVE NUMERICAL RESULTS

It can be shown that the "jump function" is given by

$$U^{sc}(x, y, +0) - U^{sc}(x, y, -0) = 2 \int_0^\infty d\nu \cos(\nu x) \int_0^\infty d\mu g(\nu, \mu) \cos(\mu y) =$$
$$-\frac{\sqrt{\pi}}{2q} H(1-\rho) \sum_{k=0}^\infty (4k+3)^{1/2} \frac{\Gamma(k+1)}{\Gamma(k+3/2)} Y_k P_{2k+1}(\sqrt{1-\rho^2}) \tag{15}$$

where the Heaviside function $H(1-\rho)$ takes the value 1 or 0 according as $\rho < 1$ or $\rho > 1$.

We conclude with some illustrative calculations for the rigid elliptic plate. The distributions of the normalised field $|U^{tot}/U_0|$ along the major axis (in normalised units $x/\lambda$) on the illuminated and shadowed sides of the elliptic plate are shown in Figure 2, left and right, respectively. Three values of the aspect ratio $q$ are examined. When $q = 1$, the plate is a circular disc and the results are in excellent agreement with those of [4] (p.547).

## CONCLUSION

An original approach to wave scattering from a thin elliptic disc is presented. In all situations the final solution is reduced to a rapidly converging i.s.l.a.e. of the second kind. Because of the Fredholm alternative each i.s.l.a.e. is solvable by the replacement of an infinite system by a finite one, i.e., solvable by a truncation method. The truncation number $N_{tr}$ must obey the simple condition $N_{tr} > kb$, where $kb$ is the relative wave number, measured along the major semi-axis $b$. The accuracy of computations is predictable and fully defined by the chosen value of $N_{tr}$. From the variety of problems to which the MoR may be applied we supplied a brief deduction for the rigid elliptic disc under normal plane wave incidence.

It is worth noting that the obtained result also highlights the way to treat similar problems for arbitrarily profiled thin plates. In fact, the chosen approach contains all principal steps to solve this class of wave scattering problems. In principle, the solution form is that of i.s.l.a.e. (9) with only the distinction being that the matrix elements contain integrands that are computed numerically.

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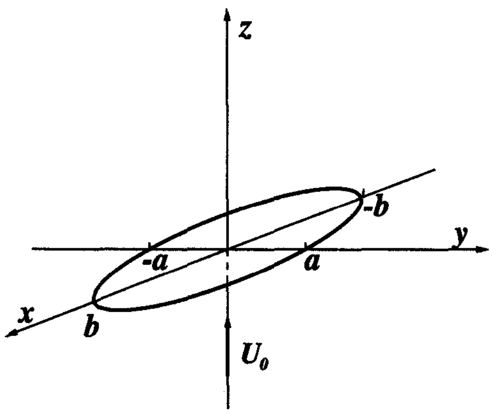


Fig. 1. The elliptic disc.

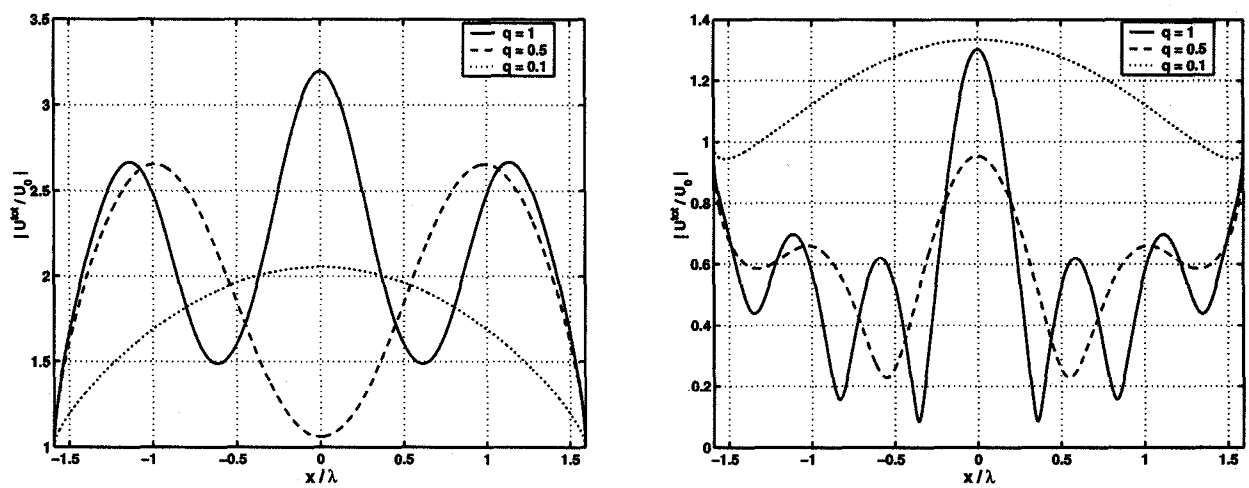


Fig. 2. Normalised field distribution on the illuminated side (left) and on the shadowed side (right) of the rigid elliptic plate, for various aspect ratios (normal incidence).